first steps

First Steps in Mathematics

Operation Sense

Operations, Computations, and Patterns and Algebra

Improving the mathematics outcomes of students

PEARSON

Department of Education and Training

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First Steps in Mathematics: Number
Operation Sense: Operations, Computations, and Patterns and Algebra

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Diagnostic Map: Number

Emergent Phase

During the Emergent Phase

Students reason about small amounts of physical materials, learning to distinguish small collections by size and recognizing increases and decreases in them. They also learn to recognize and repeat the number words used in their communities and to distinguish number symbols from other symbols. There is a growing recognition of what is the same about the way students' communities use numbers to describe collections and what is different between collections labelled with different numbers.

As a result, students come to understand that number words and symbols can be used to signify the "numerosity" of a collection.

By the end of the Emergent phase, students typically:

- use "bigger", "smaller" and "the same" to describe differences between small collections of like objects and between easily compared quantities
- anticipate whether an indicated change to a collection or quantity will make it bigger, smaller or leave it the same
- distinguish spoken numbers from other spoken words
- distinguish numerals from other written symbols
- see at a glance how many are in small collections and attach correct number names to such collections
- connect the differences they see between collections of one, two and three with the number string: "one, two, three, …"
- understand a request to share in a social sense and distribute items or portions

These students recognize that numbers may be used to signify quantity.

Matching Phase

Most students will enter the Matching phase between 3 and 5 years of age.

As students move from the Emergent phase to the Matching phase, they:

- may actually see at a glance how many there are in a small collection, such as six pebbles, yet may not be able to say the number names in order
- may say a string of the number names in order (*one, two, three, four, …*), but not connect them with how many are in collections
- may be beginning to see how to use the number names to count, but may get the order of the names wrong
- can tell by looking which of two small collections is bigger; however, they generally cannot say how much bigger
- may distribute items or portions in order to "share", but may not be concerned about whether everyone gets some, the portions are equal, or the whole amount is used up

During the Matching Phase

Students use numbers as adjectives that describe actual quantities of physical materials. Through stories, games and everyday tasks, students use one-to-one relations to solve problems where they can directly carry out or imagine the actions suggested in the situation. They learn to fix small collections to make them match, "deal out" collections or portions, and to respect the principles of counting.

As a result, students learn what people expect them to do in response to requests such as: How many are there? Can you give me six forks? How many are left? Give out one (two) each. Share them.

By the end of the Matching phase, students typically:

- recall the sequence of number names at least into double digits
- know how to count a collection, respecting most of the principles of counting
- understand that it is the last number said which gives the count
- understand that building two collections by matching one to one leads to collections of equal size, and can "fix" one collection to make it match another in size
- compare two collections one to one and use this to decide which is bigger and how much bigger
- solve small number story problems which require them to add some, take away some, or combine two amounts by imagining or role playing the situation and counting the resulting quantity
- share by dealing out an equal number of items or portions to each recipient, cycling around the group one at a time or handing out two or three at a time

These students use one-to-one relations to share and count out.

Quantifying Phase

Most students will enter the Quantifying phase between 5 and 6 or more years of age.

As students move from the Matching phase to the Quantifying phase, they:

- often do not spontaneously use counting to compare two groups in response to questions, such as: Are there enough cups for all students?
- may "skip count" but do not realize it gives the same answer as counting by ones and, therefore, do not trust it as a strategy to find how many
- often still think they could get a different answer if they started at a different place, so do not trust counting on or counting back
- often can only solve addition and subtraction problems when there is a specific action or relationship suggested in the problem situation which they can directly represent or imagine
- have difficulty linking their ideas about addition and subtraction to situations involving the comparison of collections
- may lay out groups to represent multiplicative situations, but do not use the groups to find out how many altogether, counting ones instead
- may represent division-type situations by sharing out or forming equal groups, but become confused about what to count to solve the problem, often choosing to count all the items
- may deal out an equal number of items or portions in order to share, but do not use up the whole quantity or attend to equality of the size of portions
- often do not realize that if they have shared a quantity, then counting one share will also tell them how many are in the other shares
- may split things into two portions and call them halves but associate the work "half" with the process of cutting or splitting and do not attend to equality of parts

During the Quantifying Phase

Students reason about numerical quantities and come to believe that if nothing is added to, or removed from, a collection or quantity, then the total amount must remain the same even if its arrangement or appearance is altered.

As a result, students see that the significance of the number uttered at the end of the counting process does not change with rearrangement of the collection or the counting strategy. They interpret small numbers as compositions of other numbers.

Also as a result, they develop the idea that constructing fair shares requires splitting the whole into equal parts without changing the total quantity and so begin to see the part-whole relations that link sharing and fractions.

By the end of the Quantifying phase, students typically:

- without prompting, select counting as a strategy to solve problems, such as: Are there enough cups? Who has more? Will it fit?
- use materials or visualize to decompose small numbers into parts empirically; *8 is the same as 5 with 3*
- find it obvious that when combining or joining collections counting on will give the same answer as starting at the beginning and counting the group
- make sense of the notion that there are basic facts, such as 4 + 5 is always 9, no matter how they work it out or in what arrangement
- select either counting on or counting back for subtraction problems, depending on which strategy best matches the situation
- can think of addition and subtraction situations in terms of the whole and the two parts and which is missing
- write number sentences that match how they think about the story line (semantic structure) for small number addition and subtraction problems
- realize that repeated addition or skip counting will give the same result as counting by ones
- realize that if they share a collection into a number of portions by dealing out or continuous halving and use up the whole quantity, then the portions must be equal regardless of how they look
- understand that the more portions to be made from a quantity, the smaller the size of each portion

These students use part-part-whole relations for numerical quantities.

What Is the Diagnostic Map for Number?

As students' thinking about the key mathematical concepts of Number develops, it goes through a series of characteristic phases that are described in this Diagnostic Map. Recognizing these common patterns of thinking helps teachers to interpret students' responses to activities, to understand why students seem to be able to do some things and not others and also why some students may be having difficulty in achieving certain learning goals while others may not. The Diagnostic Map also helps teachers to provide the challenges students need to move their thinking forward, to refine any half-formed ideas, and to overcome any misconceptions students may have, so they can achieve the mathematical learning goals of Number.

See over

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Diagnostic Map: Number cont.

Partitioning Phase

Most students will enter the Partitioning phase between 6 and 9 years of age.

As students move from the Quantifying phase to the Partitioning phase, they:

- often cannot decompose into parts numbers that they cannot visualize or represent as quantities, so have difficulty in partitioning larger numbers to make calculation easier; for example, students need to count forwards or backwards by ones to find the difference between 25 and 38
- often use strategies based on materials, counting on or counting back to solve addition and subtraction problems, but do not link these strategies or different problem types to a single operation (either + or -)
- may be unable to use the inverse relationship between addition and subtraction to choose the more efficient of counting on or counting back for solving particular problems
- often write their number sentences after they have solved the problem with materials, counting or basic facts, so they may be unable to write number sentences in advance when needed for problems involving larger numbers
- can count equal groups by physically or mentally laying out each group, but think of and treat each group as distinct from the others
- often believe that for two halves there must be exactly two pieces; for example, students may deny the equality of one-half and two-quarters unless the two-quarters are "stuck back together"
- although understanding that the two halves they have formed by dealing out or splitting must be equal, may think that a half formed one way could be bigger than a half formed another way
- may ignore the size of portions when choosing fraction names; for example, describing one part in seven as one-seventh regardless of whether the seven portions are equal
- often do not link sharing to unit fractions and may think that eighths are bigger than thirds because 8 is bigger than 3

During the Partitioning Phase

Students come to see the significance of whole numbers having their own meaning independent of particular countable objects. They learn to use part-whole reasoning without needing to see or visualize physical collections.

As a result, students see that numbers have magnitudes in relation to each other, can interpret any whole number as composed of two or more other numbers, and see the relationship between different types of addition and subtraction situations.

Also as a result, students see that numbers can be used to count groups and that they can use one group as a representative of other equal groups. They trust, too, that appropriate partitioning of quantities must produce equal portions.

By the end of the Partitioning phase, students typically:

- can compare whole numbers using their knowledge of the patterns in the number sequence, and think of movements between numbers without actually or mentally representing the numbers as physical quantities
- make sense of why any whole number can be rewritten as the addition of other numbers
- partition at least two- and three-digit numbers into standard component parts (*326 = 300 + 20 + 6*) without reference to actual quantities
- count up and down in tens from starting numbers like 23 or 79
- write suitable number sentences for the range of addition and subtraction situations
- use the inverse relationship between addition and subtraction to make a direct calculation possible; for example, re-interpret 43 – 27 as "what do you have to add to 27 to get 43" and so count on by tens and ones
- can double count in multiplicative situations by representing one group (*by holding up four fingers*) and counting repetitions of that same group, simultaneously keeping track of the number of groups and the number in each group
- find it obvious that two different-shaped halves from the same size whole must be the same size and are not tricked by perceptual features
- use successive splits to show that one-half is equivalent to 2 parts in 4, 4 parts in 8, and so on and expect that if the number of portions is doubled, they halve the size of each portion
- partition a quantity into a number of equal portions to show unit fractions and, given a particular quantity, will say that one-third is more than one-quarter

These students use additive thinking to deal with many-to-one relations.

Factoring Phase

Most students will enter the Factoring phase between 9 and 11 years of age.

As students move from the Partitioning phase to the Factoring phase, they:

- can "work out" a non-standard partition (*47 - 30 = 17*), but they may not see it as following automatically from the way numbers are written
- often do not realize that the digit in the tens (hundreds) place refers to groups of ten (hundred) even when they correctly use the labels "ones", "tens" and "hundreds"
- have developed ideas about decimals based on daily use for money and measures, so may think the decimal point separates two whole numbers, where the whole numbers refer to different-sized units; for example, when referring to money, they may read 6.125 as if the 6 is dollars and the 125 is cents and thus "round" it to $7.25 or say that 6.125 > 6.25
- may rightly think of decimals as another way to represent fractional numbers but, for example, think 0.6 is one-sixth
- often write related divisions and multiplications (*6 x 3 = 18, 18 ÷ 3 = 6, 18 ÷ 6 = 3*) by working each out, are unable to use the inverse relationship between division and multiplication to work out an unknown quantity
- may not understand why grouping can be used to solve a sharing problem
- can write multiplication number sentences for problems which they can think of as "groups of", but may solve other types of multiplicative problems only with materials or by counting
- do not understand why multiplication is commutative; for example, they often do not see that four piles of 13 must be the same amount as 13 groups of 4
- may believe that to show a fraction of a collection the denominator must match the total number of items and will be unable, for example, to recognize six parts in 18 as one-third
- may think of $\frac{1}{3}$ only as one part out of a collection or quantity which has been split into three equal parts, but do not also recognize it as one in each three
- may think of fractions as quantities rather than numbers and not see the significance of using the same unit as the basis for comparing fractions, so do not see why $\frac{1}{3}$ must be bigger than $\frac{1}{4}$
- may see fractions, such as three-quarters, literally as three pieces each of one-quarter and will not accept one piece which is three-quarters of the whole

During the Factoring Phase

Students extend their additive ideas about whole numbers to include the coordination of two factors needed for multiplicative thinking. They learn to construct and coordinate groups of equal size, numbers of groups and a total amount. Students also learn to visualize multiplicative situations in terms of a quantity arranged in rows and columns (an array).

As a result, students see the significance of the connection between groups of ten or groups of one hundred and the way we write whole numbers. They are able to relate different types of multiplication and division situations involving whole numbers. They also link the ideas of repeating equal groups, splitting a quantity into equal parts and fractions.

By the end of the Factoring phase, students typically:

- use their knowledge to generate alternative partitions, for example, the 2 in the tens place in 426 refers to 2 groups of 10
- sustain a correct whole number place-value interpretation in the face of conflicting information
- are flexible in their mental partitioning of whole numbers, confident that the quantity has not changed
- understand that a number can be decomposed and re-composed into its factors in a number of ways without changing the total quantity
- find it obvious that if 3 rows of 5 is 15, then both 15 divided by 3 and one-third of 15 are 5
- can visualize an array to see, for example, that five blue counters is one-third of a bag of 15 counters, both because 15 can be split into three parts each of five and one in every three counters will be blue
- visualize or draw their own diagrams to compare fractions with the same denominator ($\frac{3}{7}$ and $\frac{5}{7}$) or simple equivalences ($\frac{1}{4}$ and $\frac{2}{8}$)
- use the idea of splitting a whole into parts to understand, for example, that 2.4 is $2 + \frac{4}{10}$ and 2.45 is $2 + \frac{45}{100}$
- relate fractions and division knowing, for example, that $\frac{3}{4}$ can be thought of as $3 \div 4$ and 3 things shared among 4 students has to be $\frac{3}{4}$
- know that they can choose between multiplication or division to make calculating easier
- understand why grouping and sharing problems can be solved by the same division process
- interpret multiplication situations as "times as much" and so can see that 12 is 3 times as much as 4, and 8 is 10 times smaller than 80
- select an appropriate multiplication or division operation on whole numbers including for problems that are not easily interpreted as "groups of"; for example, combination and comparison problems
- can see why multiplication of whole numbers is commutative; for example, knowing without calculating, that 4 piles of 9 objects must be the same amount as 9 piles of 4 objects

These students think both additively and multiplicatively about numerical quantities.

Operating Phase

Most students will enter the Operating phase between 11 and 13 years of age.

As students move from the Factoring phase to the Operating phase, they:

- often continue to rely largely on their knowledge of the "named" places in reading and writing numbers, so have difficulty writing numbers with more than four digits
- may label the places to the right of the decimal point as tenths and hundredths and write 2.45 as $2 + \frac{4}{10} + \frac{5}{100}$, for example, but cannot link this with other ways of writing the decimal, such as: $2 + \frac{45}{100}$
- may think decimals with two places are always hundredths and write 2.45 as $2 + \frac{45}{100}$, but do not link this with the pattern in whole-number place value and so do not see 2.45 as $2 + \frac{4}{10} + \frac{5}{100}$
- often are unable to select a common partitioning (denominator) to enable two fractions to be compared or combined unless an equivalence they already know is involved
- often ignore the need to draw two fractions on identical wholes in order to compare or combine them
- may be unable to select an appropriate operation in situations where they cannot think of the multiplier or divisor as a whole number
- may resist selecting division where the required division involves dividing a number by a bigger number
- often believe that multiplication "makes bigger" and division "makes smaller"

During the Operating Phase

Students learn to interpret multipliers as "times as much as" or "of" rather than simply counters of groups, so can think of them as "operators" that need not be whole numbers. Students also come to see that any number can be thought of as a unit which can be repeated or split up any number of times.

As a result, students see how the intervals between whole numbers can be split and re-split into increasingly smaller intervals and realize the significance of the relationship between successive places. For example, the value of each place is ten times the value of the place to its right and one-tenth of the value of the place to its left.

Also as a result, students learn to make multiplicative comparisons between numbers, deal with proportional situations, and integrate their ideas about common and decimal fractions.

By the end of the Operating phase, students typically:

- represent common and decimal fractions both smaller and greater than 1 on a number line
- generalize their understanding of whole-number place value to include the cyclical pattern beyond the thousands, so can read, write and say any whole numbers
- use their understanding of the relationship between successive places to order decimal numbers regardless of the number of places
- use the cyclical pattern in the places to count forwards and backwards in tenths, hundredths, thousandths, including up and over whole numbers
- are flexible in partitioning decimal numbers
- realize that for multipliers smaller than 1, multiplication makes smaller, and for divisors smaller than 1, division makes bigger
- select an appropriate number of partitions to enable a quantity to be shared into two different numbers of portions; such as 5 or 3
- construct successive partitions to model multiplication situations; *I took half the cake home and then ate one-third of it*
- produce their own diagrams to compare or combine two fractions, ensuring that both fractions ($\frac{2}{3}$ and $\frac{1}{4}$) are represented on identical wholes
- split and recombine fractions visually or mentally to add or subtract; $\frac{1}{2} + \frac{1}{4}$ is $(\frac{1}{4}, \frac{1}{4}) + \frac{1}{4} = \frac{3}{4}$
- recognize the need to multiply in situations where the multiplier is a fractional number
- can write suitable number sentences for the full range of multiplication and division situations involving whole numbers, decimals and fractions

These students can think of multiplications and divisions in terms of operators.

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INTRODUCTION

The *First Steps in Mathematics* resource books and professional development program are designed to help teachers plan, implement and evaluate the mathematics curriculum they provide for students. The series describes the key mathematical ideas students need to understand in order to achieve the principal learning goals of mathematics curricula across Canada and around the world.

Unlike many resources that present mathematical concepts that have been logically ordered and prioritized by mathematicians or educators, *First Steps in Mathematics* follows a sequence derived from the mathematical development of real children. Each resource book is based on five years of research by a team of teachers from the Western Australia Department of Education and Training, and tertiary consultants led by Professor Sue Willis at Murdoch University.

The *First Steps in Mathematics* project team conducted an extensive review of international research literature, which revealed gaps in the field of knowledge about students' learning in mathematics. Many of these findings are detailed in the Background Notes that supplement the Key Understandings described in the *First Steps in Mathematics* resource books for Number.

Using tasks designed to replicate those in the research literature, team members interviewed hundreds of elementary school children in diverse locations. Analysis of the data obtained from these interviews identified characteristic phases in the development of students' thinking about mathematical concepts.

The Diagnostic Maps—which appear in the resource books for Number, Measurement, Geometry and Space, and Data Management and Probability—describe these phases of development, exposing specific markers where students often lose, or never develop, the connection between mathematics and meaning. Thus, *First Steps in Mathematics* helps teachers systematically observe not only what mathematics individual children do, but how the children do the mathematics, and how to advance the children's learning.

It has never been more important to teach mathematics well. Globalization and the increasing use of technology have created changing demands for the application of mathematics in all aspects of our lives. Teaching mathematics well to all students requires a high level of understanding of teaching and learning in mathematics and of mathematics itself. The *First Steps in Mathematics* series and professional development program help teachers provide meaningful learning experiences and enhance their capacity to decide how best to help all students achieve the learning goals of mathematics.

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Chapter 1

An Overview of *First Steps in Mathematics*

First Steps in Mathematics is a professional development program and series of teacher resource books that are organized around mathematics curricula for Number, Measurement, Geometry and Space, and Data Management and Probability.

The aim of *First Steps in Mathematics* is to improve students' learning of mathematics.

First Steps in Mathematics examines mathematics within a developmental framework to deepen teachers' understanding of teaching and learning mathematics. The developmental framework outlines the characteristic phases of thinking that students move through as they learn key mathematical concepts. As teachers internalize this framework, they make more intuitive and informed decisions around instruction and assessment to advance student learning.



First Steps in Mathematics helps teachers to:

- build or extend their own knowledge of the mathematics underpinning the curriculum
- understand how students learn mathematics so they can make sound professional decisions
- plan learning experiences that are likely to develop the mathematics outcomes for all students
- recognize opportunities for incidental teaching during conversations and routines that occur in the classroom

This chapter details the beliefs about effective teaching and learning that *First Steps in Mathematics* is based on and shows how the elements of the teacher resource books facilitate planning and instruction.

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Beliefs about Teaching and Learning

Focus Improves by Explicitly Clarifying Outcomes for Mathematics

Learning is improved if the whole-school community has a shared understanding of the mathematics curriculum goals, and an implementation plan and commitment to achieving them. A common understanding of these long-term aims helps individuals and groups of teachers decide how best to support and nurture students' learning, and how to tell when this has happened.

All Students Can Learn Mathematics to the Best of Their Ability

A commitment to common goals signals a belief that **all** students can be successful learners of mathematics. A situation where less is expected of and achieved by certain groups of students is not acceptable. School systems, schools and teachers are all responsible for ensuring that **each** student has access to the learning conditions he or she requires to achieve the curricular goals to the best of his or her ability.

Learning Mathematics Is an Active and Productive Process

Learning is not simply about the transfer of knowledge from one person to another. Rather, students need to construct their own mathematical knowledge in their own way and at a pace that enables them to make sense of the mathematical situations and ideas they encounter. A developmental learning approach is based on this notion of learning. It recognizes that not all students learn in the same way, through the same processes, or at the same rate.

Common Curricular Goals Do Not Imply Common Instruction

The explicit statement of the curricular goals expected for all students helps teachers to make decisions about the classroom program. However, the list of content and process goals for mathematics is not a curriculum. If all students are to succeed to the best of their ability on commonly agreed concepts, different curriculum implementations will not only be possible, but also be necessary. Teachers must decide what type of instructional activities are needed for their students to achieve the learning goals.

> *A curriculum that enables all students to learn must allow for different starting points and pathways to learning so that students are not left out or behind.*
>
> **—Darling-Hammond, *National standards and assessments*, p. 480**

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Professional Decision-Making Is Central in Teaching

It is the responsibility of teachers to provide all students with the conditions necessary for them to achieve the curricular goals of mathematics. This responsibility requires teachers to make many professional decisions simultaneously, such as what to teach, to whom, and how, and making these professional decisions requires a synthesis of knowledge, experience, and evidence.

> *Professionalism has one essential feature; …(it) requires the exercise of complex, high level professional judgments…(which) involve various mixes of specialised knowledge; high level cognitive skills; sensitive and sophisticated personal skills; broad and relevant background and tacit knowledge.*
>
> **—Preston, *Teacher professionalism*, p. 2, 20**

The personal nature of each student's learning journey means that the decisions teachers make are often "non-routine", and the reasoning processes involved can be complex. These processes cannot be reduced to a set of instructions about what to do in any given situation. Teachers must have the freedom and encouragement to adapt existing curricula flexibly to best meet their students' needs and to move them forward. The improvement of students' learning is most likely to take place when teachers have good information about tasks, response range and intervention techniques on which to base their professional decisions.

"Risk" Relates to Future Mathematics Learning



Risk cannot always be linked directly to students' current achievement. Rather, it refers to the likelihood that their future mathematical progress is "at risk".

Some students who can answer questions correctly might not have the depth of understanding needed for ongoing progress. Others might have misconceptions that could also put their future learning "at risk". A number of students may make errors that are common when they try to make sense of new mathematical ideas and, therefore, show progress. For example, a student who writes "six hundred four" as "6004" is incorrect. However, this answer signals progress because the student is using his or her knowledge of the fact that the hundreds are written with two zeros.

Students who are learning slowly, or whose previous experiences are atypical, might nevertheless progress steadily if their stage of learning is accommodated with appropriate, but challenging, learning experiences.

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Successful Mathematics Learning Is Robust Learning

Robust learning, which focuses on students developing mathematics concepts fully and deeply, is essential if learning is to be sustained over the long term.

A focus on short-term performance or procedural knowledge at the expense of robust knowledge places students "at risk" of not continuing to progress throughout the years of schooling.

Learning Mathematics: Implications for the Classroom

Learning mathematics is an active and productive process on the part of the learner. The following section illustrates how this approach influences the ways in which mathematics is taught in the classroom.

Learning Is Built on Existing Knowledge

Learners' interpretations of mathematical experiences depend on what they already know and understand. For example, many young students start school with the ability to count collections of seven or eight objects by pointing and saying the number names in order. However, they may not have the visual memory to recognize seven or eight objects at a glance. Others may readily recognize six or seven objects at a glance without being able to say the number names in order.

In each case, students' existing knowledge should be recognized and used as the basis for further learning. Their learning should be developed to include the complementary knowledge, with the new knowledge being linked to and building on students' existing ideas.



Learning Requires That Existing Ideas Be Challenged

Learning requires that students extend or alter what they know as a result of their knowledge being challenged or stretched in some way. For example, a challenge may occur when a student predicts that the tallest container will hold the most water, then measures and finds that it does not.

Another challenge may occur when a student believes that multiplication makes numbers bigger and then finds that this is not true for some numbers. Or, it may happen when the student finds that peers think about a problem in a different way. The student must find some way of dealing with the challenge or conflict provided by the new information in order to learn.

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Learning Occurs when the Learner Makes Sense of the New Ideas

Teaching is important—but learning is done *by* the learner rather than *to* the learner. This means the learner acts on and makes sense of new information. Students almost always try to do this. However, in trying to make sense of their mathematical experiences, some students will draw conclusions that are not quite what their teachers expect.

Also, when students face mathematical situations that are not meaningful, or are well beyond their current experience and reach, they often conclude that the mathematics does not make sense or that they are incapable of making sense of it. This may encourage students to resort to learning strategies based on the rote imitation of procedures. The result is likely to be short-term rather than effective long-term learning. Teachers have to provide learning experiences that are meaningful and challenging, but within the reach of their students.

Learning Involves Taking Risks and Making Errors

In order to learn, students have to be willing to try a new or different way of doing things, and stretch a bit further than they think they can. At times, mistakes can be a sign of progress. For example, students often notice that each number place, from right to left, has a new name—ones, tens, hundreds and thousands—but they may predict incorrectly that the next place will be millions. Such errors can be a positive sign that students are trying to generalize the patterns in the way we write numbers.

Errors can provide a useful source of feedback, challenging students to adjust their conceptions before trying again. Errors may also suggest that learners are prepared to work on new or difficult problems where increased error is likely. Or, they may try improved ways of doing things that mean giving up old and safe, but limited, strategies. For example, a student who can successfully find "five twenty-sixes" by adding the number 26 five times takes a risk when trying to do it by multiplying, since multiplying may result in increased mistakes in the short term.

Learners Get Better with Practice

Students should get adequate opportunities to practise mathematics, but this involves much more than the rote or routine repetition of facts and procedures. For example, if students are to learn how to plan data collection, they will need plenty of opportunities to actually plan their own surveys and experiments, note for themselves when things do not work as expected, and improve their collection processes to improve their data.

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Likewise, if students are to develop good mental arithmetic, they will need spaced and varied practice with a repertoire of alternative addition strategies and with choosing among them. Extensive repetitive practice on a single written addition algorithm is unlikely to help with this. In fact, it is more likely to interfere with it.

Learning Is Helped by Clarity of Purpose for Students as well as Teachers

Learning is likely to be enhanced if students understand what kind of learning activity they should be engaged in at any particular time. This means helping students to distinguish between tasks that provide practice of an already learned procedure and tasks that are intended to develop understanding of mathematical concepts and processes. In the former case, little that is new is involved, and tasks are repetitive, so they become habitual and almost unthinking. Students should expect to be able to start almost immediately and, if they cannot, realize that they may need to know more and seek help.

With tasks that are intended to develop understanding, non-routine tasks and new ideas may be involved. Students should not expect to know what to do or to be able to get started immediately.

Students may spend a considerable amount of time on a single task, and they will often be expected to work out for themselves what to do. They should recognize that, for such activities, persistence, thoughtfulness, struggle and reflection are expected.

Teaching Mathematics

Teachers assume considerable responsibility for creating the best possible conditions for learning. The kind of learning tasks and environment teachers provide depend on their own view of how learning is best supported. The perspective that learning is an active and productive process has two significant implications for teaching.

First, teachers cannot predict or control exactly what and when students learn. They need to plan curricula that provide students with a wider and more complex range of information and experiences than they would be expected to understand fully at any given time. For example, using the constant function on their calculators, grade 1 students may be able to "count" into the thousands. Their teacher may encourage students' exploration, read the numbers for them and stimulate their curiosity about large numbers in general. This enables students to begin developing notions about counting and numbers at many different levels. However, their teacher may only *expect* them to demonstrate a full understanding of the connection between quantity and counting for small numbers.

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This represents a significant change in curriculum planning. It is a movement away from an approach that only exposes students to content and ideas that they should be able to understand or do at a particular point in time.

Second, for students to become effective learners of mathematics, they must be engaged fully and actively. Students will want, and be able, to take on the challenge, persistent effort and risks involved. Equal opportunities to learn mathematics means teachers will:

- provide an environment for learning that is equally supportive of all students
- offer each student appropriate mathematical challenges
- foster in all students processes that enhance learning and contribute to successful achievment of goals

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Understanding the Elements of *First Steps in Mathematics*

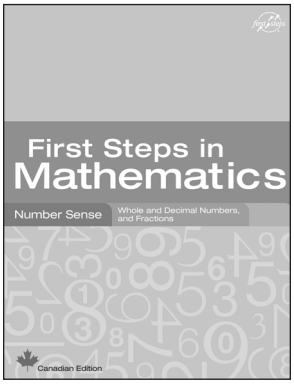
The elements of *First Steps in Mathematics* embody the foregoing beliefs about teaching and learning and work together to address three main questions:

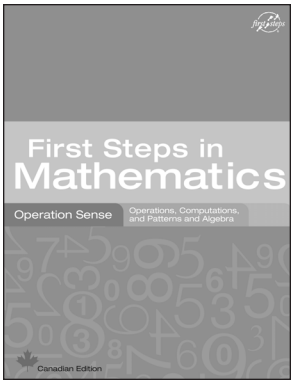
- What are students expected to learn?
- How does this learning develop?
- How do teachers advance this learning?

Learning Outcomes for the Number Strand

The Number strand in *First Steps in Mathematics* focuses on numbers and operations—what they mean, how we represent them, and how and why we use them in our everyday lives. The overall goal for this strand is to help students develop a flexible sense of numbers and operations and the relationships between them. Students need to develop confidence in their ability to deal with numerical situations with flexibility, ease and efficiency.

To achieve this, students require a deep understanding of the meanings of numbers and how we write them. They also need to develop reflective knowledge about the meaning and use of basic operations, a working and flexible repertoire of computational skills, and the capacity to identify and work with number patterns and relationships. A wide range of learning experiences can enable students to comprehend numbers, understand operations, and calculate and apply reasoning skills to number patterns and relationships.





The *First Steps in Mathematics* resource books for Number examine five outcomes essential for mathematical literacy. These outcomes describe the learning expectations for students and the goals of instruction:

Whole and Decimal Numbers

Read, write and understand the meaning, order and relative magnitudes of whole and decimal numbers, moving flexibly between equivalent forms.

Fractions

Read, write and understand the meaning, order and relative magnitudes of fractional numbers, moving flexibly between equivalent forms.

Operations

Understand the meaning, use and connections between addition, multiplication, subtraction and division.

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Computations
Choose from a repertoire of mental, paper and calculator computational strategies for each operation and apply appropriately to meet the required degrees of accuracy and judge the reasonableness of results.

Patterns and Algebra
Investigate, generalize and reason about patterns in numbers, explaining and justifying the conclusions reached.

Integrating the Outcomes
Each mathematics outcome in Number is explored in a separate chapter of the resource books. This is to emphasize both the importance of each outcome and the differences between them. For example, students need to learn about the meaning, properties and use of addition (Operations) as well as being able to add numbers (Computations). By paying separate and special attention to each outcome, teachers can make sure that both areas receive sufficient attention, and that the important ideas about each are drawn out of the learning experiences they provide.

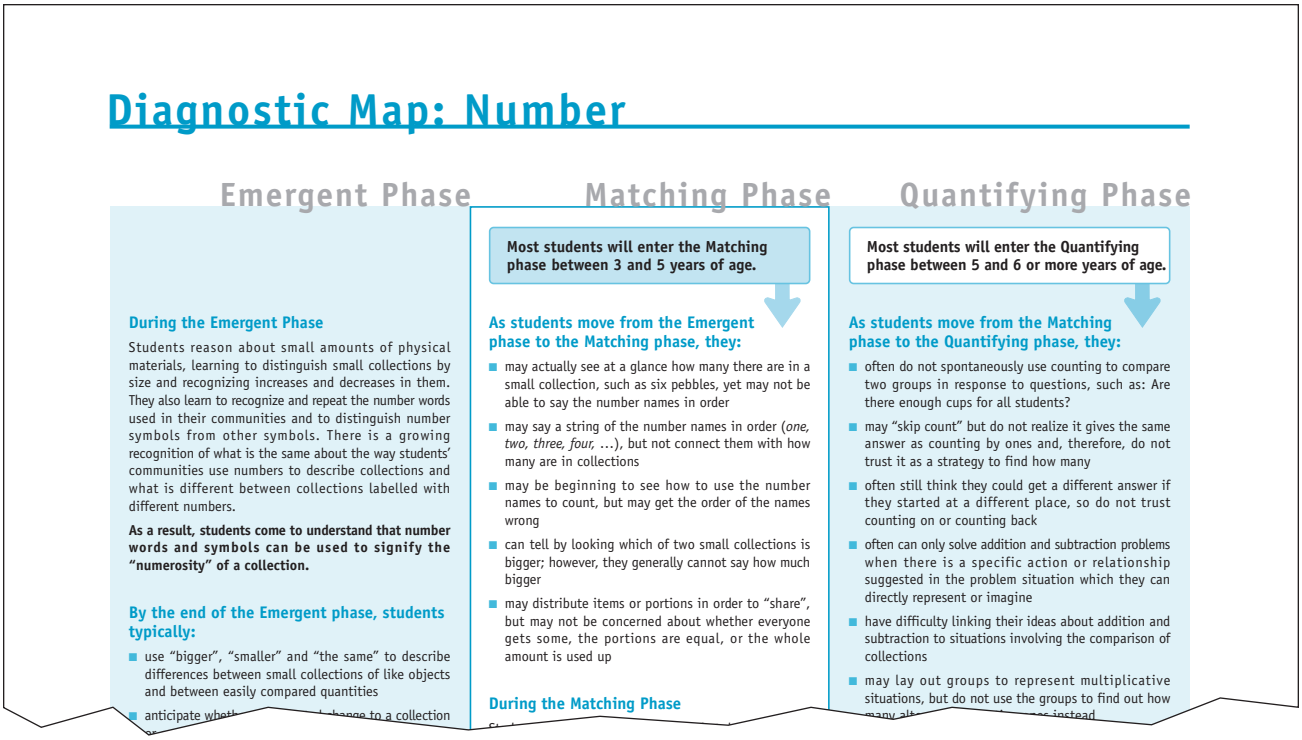
This does not mean, however, that the ideas and skills underpinning each of the outcomes should be taught separately, or that they will be learned separately. The learning goals are inextricably linked. Consequently, many of the activities will provide opportunities for students to develop their ideas about more than one of the outcomes. This will help teachers to ensure that the significant mathematical ideas are drawn from the learning activities, so that their students achieve each of the mathematics curriculum expectations for Number.

How Does This Learning Develop?
First Steps in Mathematics: Number describes characteristic phases in students' thinking about the major mathematical concepts of the Number strand. These developmental phases are organized in a Diagnostic Map.

Diagnostic Map
The Diagnostic Map for Number details six developmental phases. It helps teachers to:

- understand why students seem to be able to do some things and not others
- realize why some students may be experiencing difficulty while others are not
- indicate the challenges students need to move their thinking forward, to refine their preconceptions, overcome any misconceptions, and so develop deep reflective understanding about concepts
- interpret students' responses to activities

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The Diagnostic Map includes key indications and consequences of students' understanding and growth. This information is crucial for teachers making decisions about their students' level of understanding of mathematics. It enhances teachers' decisions about what to teach, to whom and when to teach it.

Each developmental phase of the Diagnostic Map has three aspects. The first aspect describes the learning focus *during the phase*. The second aspect details typical thinking and behaviours of students *by the end of the phase*. The third outlines preconceptions, partial conceptions or misconceptions that may still exist for students at the end of the phase. This aspect provides the learning challenges and teaching emphases *as students move to the next phase*.

Diagnostic Tasks

First Steps in Mathematics: Number provides a series of short, focused Diagnostic Tasks in the *Course Book*. These tasks have been validated through extensive research with students and help teachers locate individual students on the Diagnostic Map.

How Do Teachers Advance This Learning?

To advance student learning, teachers identify the big mathematical ideas, or key understandings, of the outcomes, or curricular goals. Teachers plan learning activities to develop these key understandings. As learning activities provide students with opportunities and support to develop new insights, students begin to move to the next developmental phase of mathematical thinking.

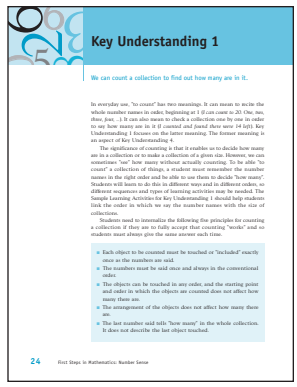
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Key Understandings

The Key Understandings are the cornerstone of *First Steps in Mathematics*. The Key Understandings:

- describe the mathematical ideas, or concepts, which students need to know in order to achieve curricular goals
- explain how these mathematical ideas form the underpinnings of the mathematics curriculum statements
- suggest what experiences teachers should plan for students so that they move forward in a developmentally appropriate way
- provide a basis for the recognition and assessment of what students already know and still need to know in order to progress along the developmental continuum and deepen their knowledge
- indicate the emphasis of the curriculum at particular stages
- provide content and pedagogic advice to assist with planning the curriculum at the classroom and whole-school levels

The number of Key Understandings for each mathematics curricular goal varies according to the number of "big mathematical ideas" students need to achieve the goal.


Key Understanding 1

We can count a collection to find out how many are in it.

In everyday use, "to count" has two meanings. It can mean to recite the whole number names in order, beginning at 1 (I can count to 20. One, two, three, four, ...). It can also mean to check a collection one by one in order to say how many are in it (I counted and found there were 14 left). Key Understanding 1 focuses on the latter meaning. The former meaning is an aspect of Key Understanding 4.

The significance of counting is that it enables us to decide how many are in a collection or to make a collection of a given size. However, we can sometimes "see" how many without actually counting. To be able "to count" a collection of things, a student must remember the number names in the right order and be able to use them to decide "how many". Students will learn to do this in different ways and in different orders, so different sequences and types of learning activities may be needed. The Sample Learning Activities for Key Understanding 1 should help students link the order in which we say the number names with the size of collections.

Students need to internalize the following five principles for counting a collection if they are to fully accept that counting "works" and so students must always give the same answer each time.

- Each object to be counted must be touched or "included" exactly once as the numbers are said.
- The numbers must be said once and always in the conventional order.
- The objects can be touched in any order, and the starting point and order in which the objects are counted does not affect how many there are.
- The arrangement of the objects does not affect how many there are.
- The last number said tells "how many" in the whole collection. It does not describe the last object touched.

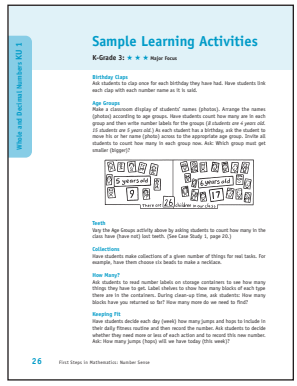
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Sample Learning Activities

For each Key Understanding, there are Sample Learning Activities that teachers can use to develop the mathematical ideas of the Key Understanding. The activities are organized into three broad groups:

- activities suitable for students in Kindergarten to Grade 3
- activities for students in Grades 3 to 5
- activities for students in Grades 5 to 8

If students in Grades 5 to 8 have not had enough prior experience, then teachers may need to select and adapt activities from earlier groups.


Whole and Decimal Numbers: KU 1

Sample Learning Activities

K-Grade 3: ★ ★ ★ Major Focus

Birthday Claps

Ask students to clap once for each birthday they have had. Have students link each clap with each number name as it is said.

Age Groups

Make a classroom display of students' names (photos). Arrange the names (photos) according to age groups. Have students count how many are in each group and then write number labels for the groups (*If students are 4 years old, 15 students are 5 years old.*) As each student has a birthday, ask the student to move his or her name (photo) across to the appropriate age group. Invite all students to count how many in each group now. Ask: Which group must get smaller (bigger)?

Teeth

Vary the Age Groups activity above by asking students to count how many in the class have (have not) lost teeth. (See Case Study 1, page 20.)

Collections

Have students make collections of a given number of things for real tasks. For example, have them choose six beads to make a necklace.

How Many?

Ask students to read number labels on storage containers to see how many things they have to get. Label shelves to show how many blocks of each type there are in the containers. During clean-up time, ask students: How many blocks have you returned so far? How many more do we need to find?

Keeping Fit

Have students decide each day (week) how many jumps and hops to include in their daily fitness routine and then record the number. Ask students to decide whether they need more or less of each action and to record this new number. Ask: How many jumps (hops) will we have today (this week)?

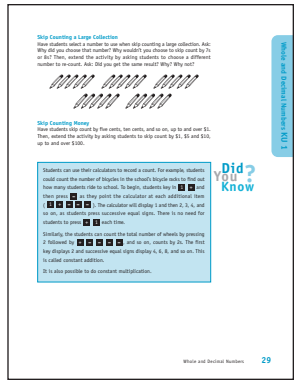
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Case Studies

The Case Studies illustrate some of the ways in which students have responded to Sample Learning Activities. The emphasis is on how teachers can focus students' attention on the mathematics during the learning activities.

"Did You Know?" Sections

For some of the Key Understandings, there are "Did You Know?" sections. These sections highlight common understandings and misunderstandings that students have. Some "Did You Know?" sections also suggest diagnostic activities that teachers may wish to try with their students.


Skip Counting a Large Collection

Have students select a number to use when skip counting a large collection. Ask: Why did you choose that number? Why wouldn't you choose to skip count by 7s or 8s? Then, extend the activity by asking students to choose a different number to re-count. Ask: Did you get the same result? Why? Why not?

Skip Counting Money

Have students skip count by five cents, ten cents, and so on, up to and over $1. Then, extend the activity by asking students to skip count by $1, $5 and $10, up to and over $100.

Students can use their calculators to record a count. For example, students could count the number of bicycles in the school's bicycle rack or find out how many students ride to school. To begin, students key in [0] [+] and then press [+] as they point the calculator at each additional item [1] [+] [+] [+] [+]. The calculator will display 1 and then 2, 3, 4, and so on, as students press successive equal signs. There is no need for students to press [1] each time.

Similarly, the students can record the total number of wheels by pressing 2 followed by [+] [+] [+] [+] [+] and so on, counts by 2s. The first key displays 2 and successive equal signs display 4, 6, 8, and so on. This is called constant addition.

It is also possible to do constant multiplication.

Did You Know?

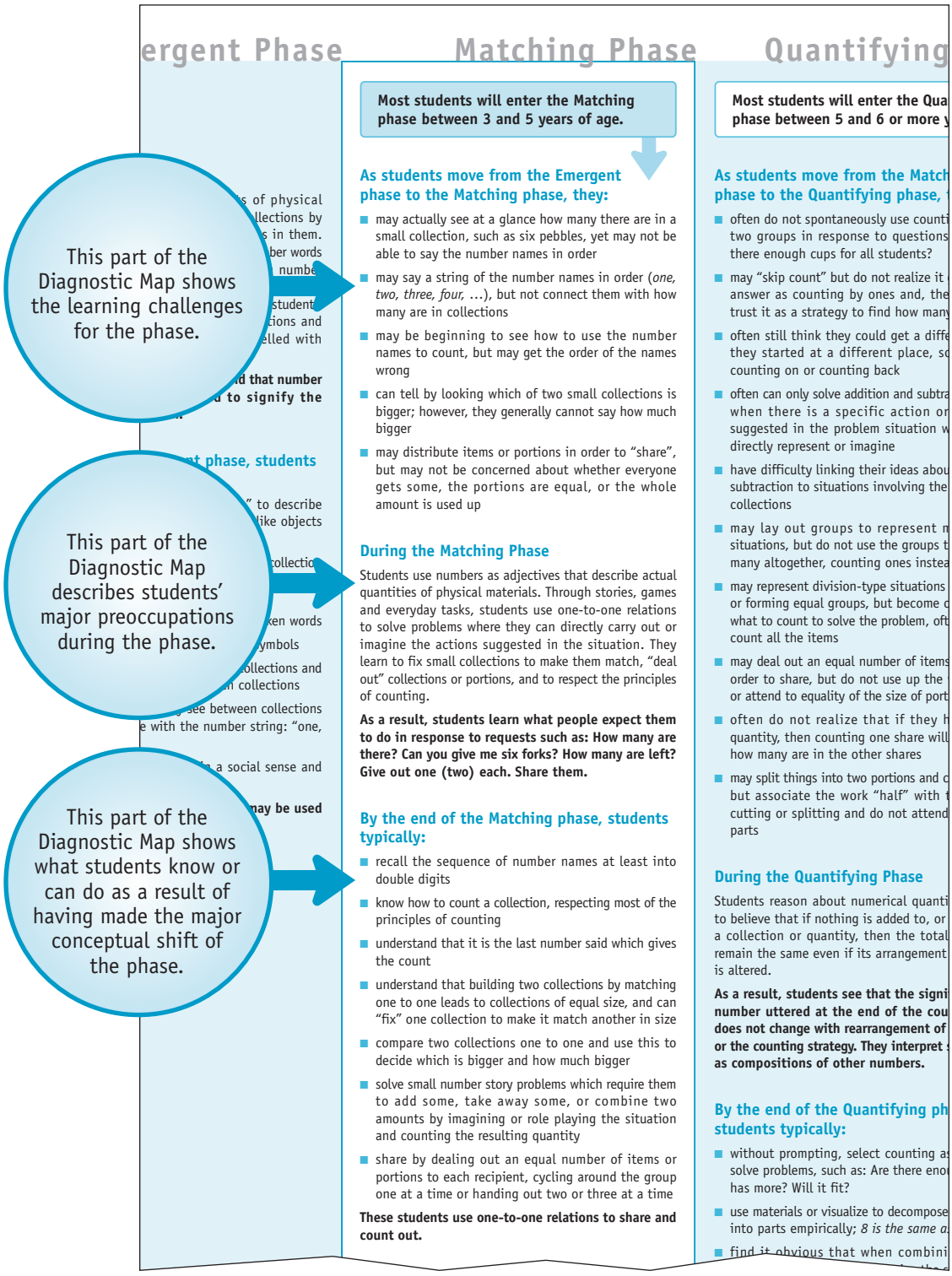
Whole and Decimal Numbers: KU 1

Whole and Decimal Numbers 29

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How to Read the Diagnostic Map

The Diagnostic Map for Number has six phases: Emergent, Matching, Quantifying, Partitioning, Factoring and Operating. The diagram on this page shows the second phase, the Matching phase.



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The text in the "During the phase" section describes students' major preoccupations, or focus, during that phase of thinking about Number.

The "By the end" section of each phase provides examples of what students typically think and are able to do as a result of having worked through the phase.

The achievements described in the "By the end" section should be read in conjunction with the "As students move" section. Together, these two sections illustrate that although students might have developed a range of important understandings as they passed through the phase, they might also have developed some unconventional or unhelpful ideas. Both of these sections of the Diagnostic Map are intended as a useful guide only. Teachers will recognize more examples of similar thinking in the classroom.

How Do Students Progress Through the Phases?

Students who have passed through one phase of the Diagnostic Map are entering the next phase. They bring behaviours and understandings from one phase to the next. For example, the text in the "As students move from the Emergent Phase" section describes the behaviours students bring to the Matching phase. This section includes the preconceptions, partial conceptions and misconceptions that students may have developed along the way. These provide the learning challenges for the next phase.

Linking the Diagnostic Maps and Learning Goals

Students are unlikely to achieve full conceptual understanding unless they have moved through certain phases of the Diagnostic Map. However, passing through the phase does not guarantee that the concept has been mastered. Students might have the conceptual development necessary for deepening their understanding, but without access to a curriculum that enables them to learn the necessary foundation concepts described in a particular phase, they will be unable to do so.

The developmental phases help teachers interpret students' responses in terms of pre- and partial conceptions. If, for example, a student cannot count on, despite a teacher spending considerable time with that student, then the phases can help explain what the problem is. In this case, the student may not be through the Quantifying phase for Number and so may not trust the count. No amount of practice or telling the student to "hold the number in your head" will help. The source of the problem is that the student does not trust that the initial quantity remains the same. This concept must be developed before the student can learn to count on.

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How Will Teachers Use the Diagnostic Map?

The Diagnostic Map is intended to assist teachers as they plan for mathematics teaching and learning. The descriptions of the phases help teachers make informed decisions about students' understandings of the mathematical concepts. The map will help teachers understand why students can do some things and not others, and why some students may be having difficulty achieving certain outcomes.

Initially, teachers may use the Diagnostic Map to extend their own knowledge about how students typically learn mathematics. Knowing about the major conceptual shifts in each phase and their links to mathematical learning goals will help teachers decide which Key Understandings should be the major focus for classroom planning.

Familiarity with the behaviours described in the phases will enhance the informed decisions teachers make about what they observe students doing and saying during lessons. The information obtained over time about the major preoccupations of students informs ongoing planning. As teachers begin to understand the typical behaviours of each phase, this planning process will become more efficient.



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Planning with *First Steps in Mathematics*

Using Professional Decision-Making to Plan

The *First Steps in Mathematics* resource books and professional development support the belief that teachers are in the best position to make informed decisions about how to help their students achieve conceptual understanding in mathematics. Teachers will base these decisions on knowledge, experience and evidence.

The process of using professional decision-making to plan classroom experiences for students is fluid, dependent on the situation and context, and varies from teacher to teacher. The selection of learning activities and appropriate focus questions will be driven by each teacher's knowledge of his or her students and their learning needs, the mathematics, and mathematics-related pedagogy. The *First Steps in Mathematics* resource books and professional development focus on developing this pedagogical content knowledge.

The diagram on the next page illustrates how these components combine to inform professional decision-making. There is no correct place to start or finish, or process to go through. Circumstances and experience will determine both the starting point and which component takes precedence at any given time.

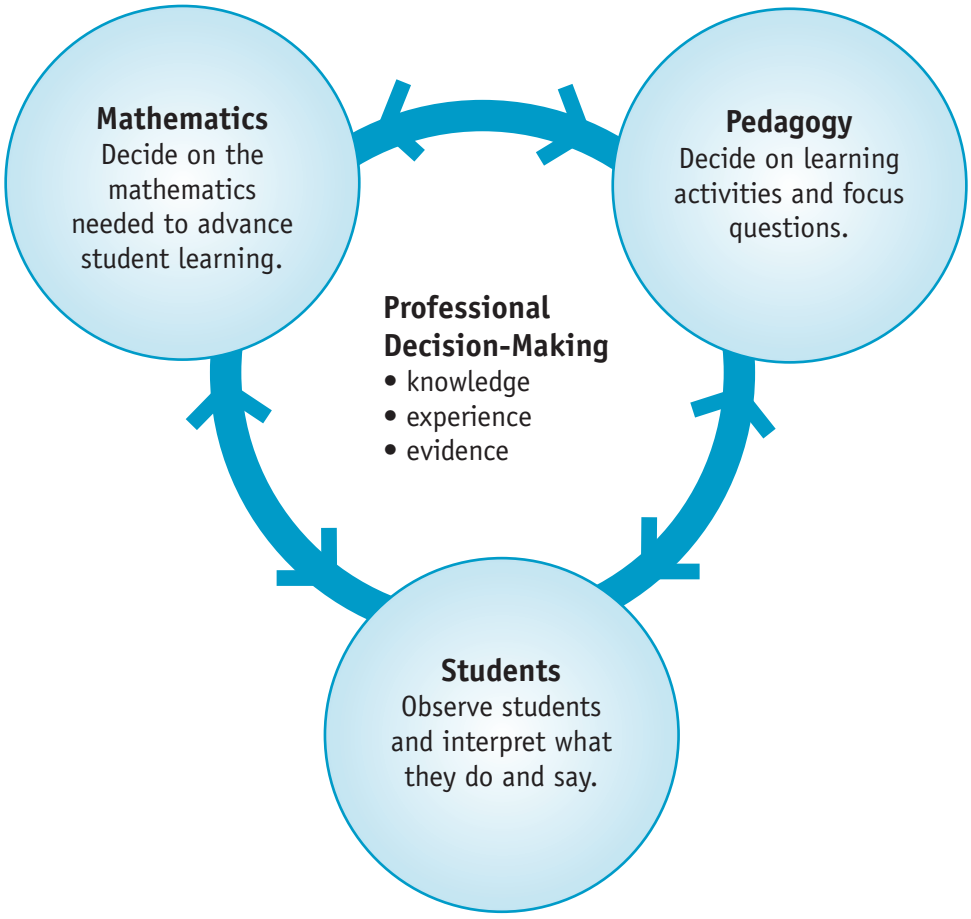
Different teachers working with different students may make different decisions about what to teach, to whom, when and how.



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The process is about selecting activities that enable all students to learn the mathematics described in curriculum focus statements. More often than not, teachers' choice of activities and focus questions will be driven by their knowledge of their students and the mathematics. At other times, teachers might select an activity to help them assess students' existing knowledge or because of the specific mathematics in the task. Whatever the starting point, the *First Steps in Mathematics* resource books and professional development will help teachers to ensure that their mathematics pedagogy is well informed.

The examples on the opposite page show some of the different ways teachers can begin planning using *First Steps in Mathematics*.



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Focusing on the Mathematics

Teachers may choose to focus on the mathematics, deciding on the mathematics they think they need to move students on.

What mathematics do my students need to know?

Mathematics
Decide on the mathematics needed to advance student learning.

What sections of ***First Steps in Mathematics*** do I look at?

- Key Understandings and Key Understandings descriptions

Understanding What Students Already Know

Teachers may choose to start by finding out what mathematics their students already know.

What do my students know about these mathematics concepts?

Students
Observe students and interpret what they do and say.

What sections of ***First Steps in Mathematics*** do I look at?

- Key Understandings and Key Understandings descriptions
- "Did You Know?" sections
- Diagnostic Map
- Diagnostic Tasks

Developing Students' Knowledge

Teachers may begin by planning and implementing some activities to develop their knowledge of students' learning.

What activities will help my students develop these ideas? How will I draw out the mathematical ideas from the learning activity?

Pedagogy
Decide on learning activities and focus questions.

What sections of ***First Steps in Mathematics*** do I look at?

- Sample Learning Activities
- Case Studies
- Key Understandings and Key Understandings descriptions

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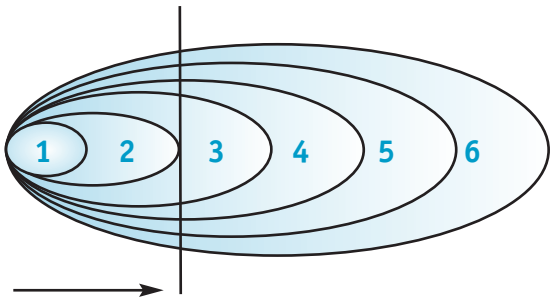
Planning

The mathematics curriculum goals and developmental phases described in the Diagnostic Map help teachers to know where students have come from and where they are heading. This has implications for planning. While day-to-day planning may focus on the mathematics of particular Key Understandings, teachers must keep in mind the learning required for progression through the developmental phases.

If a student has reached the end of the Matching phase, then the majority of experiences the teacher provides will relate to reaching the end of the Quantifying phase. However, some activities will also be needed that, although unnecessary for reaching the Quantifying phase, will lay important groundwork for reaching the Partitioning phase and even the Factoring phase.

For example, students do not typically understand the inverse relationship between addition and subtraction until the middle years of elementary school. Therefore, using the inverse relationship between addition and subtraction to solve problems is not expected for reaching the end of the Quantifying phase, but it is for reaching the end of Partitioning phase. Given access to an appropriate curriculum in Number, most students should be able to reach the Partitioning phase, selecting appropriate operations to solve problems, by the end of the middle years of elementary school. If students are to develop these ideas in a timely manner, then they cannot be left until after reaching the end of the Quantifying phase.

Developmental Phases



There are a number of reasons for this approach. First, it is anticipated that a considerable number of students will enter the middle years of elementary school having reached the end of the Partitioning phase. Second, if teachers were to wait until the middle years to start teaching about the inverse relationship between addition and subtraction, then it is unlikely that students would develop all the necessary concepts and skills in one year.

Third, work in the middle years of elementary school should not only focus on the Partitioning phase, but also provide the groundwork for students to reach the Factoring phase in the next year or two, and the Operating phase some time later.

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Teachers, who plan on the basis of deepening the understanding of the concepts, would think about the expected long-term learning in the early years of schooling. They would provide experiences that lead to the learning goals at the Partitioning and Factoring phases. This means students may be challenged about the significance of the inverse relationship between addition and subtraction in simple contexts during the first few years of schooling. They may not yet be ready to use the inverse relationship between addition to solve particular problems. It will take several years of learning experiences in a variety of contexts to culminate in a full understanding.

Monitoring Students over Time

By describing progressive conceptual development that spans the elementary-school years, teachers can monitor students' individual long-term mathematical growth as well as their long-term progress against an external standard. This long-term monitoring is one of the reasons why a whole-school approach is essential. For example, Sarah, has reached the end of the Factoring phase for each of the Number concepts while another student, Maria, has only just reached the Matching phase.

By comparing Maria and Sarah's levels against the standard, their teacher is able to conclude that Sarah is progressing well, but Maria is not. This prompts Maria's teacher to investigate Maria's thinking about Number and to plan specific support.

However, if two years later, Sarah has not reached the end of the Operating phase while Maria has reached the end of the Partitioning phase and is progressing well towards reaching the Factoring phase, they would both now be considered "on track" against an external standard. Sarah's achievement is more advanced than Maria's, but in terms of individual mathematical growth, Sarah appears to have stalled. Her progress may now be of greater concern than Maria's.

Reflecting on the Effectiveness of Planned Lessons

The fact that activities were chosen with particular mathematical learning goals in mind does not mean that they will have the desired result. Sometimes, students deal with an activity successfully, but they use different mathematics than teachers anticipated. Different activities related to the mathematics that has not been learned may need to be provided in the future.

On other occasions, what students actually learn may not be what teachers intended them to learn. Students may surprise teachers and cause them to rethink the activity.

In some instances, activities, which teachers think will help students develop particular mathematical ideas, do not generate those ideas. This can occur even when students complete the activity as designed.

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The evidence about what students are actually thinking and doing during their learning experiences is the most important source of professional learning and decision-making. At the end of every activity, teachers need to ask themselves: *Have the students learned what was intended for this lesson? If not, why not?* These questions are at the heart of improving teaching and learning. Teachers make constant professional, informed evaluations about whether the implemented curriculum is resulting in the intended learning goals for students. If not, then teachers need to change the experiences provided.

Teachers' decisions, when planning and adjusting learning activities as they teach, are supported by a clear understanding of:

- the desired mathematics conceptual goal of the selected activities
- what progress in mathematics looks like
- what to look for as evidence of students' deepening understanding

When planning day-to-day lessons, it is important for teachers to appreciate that many of the same activities will be appropriate for students who are working within a range of developmental phases. Teachers can accommodate the differences in understanding and development among students by:

- asking different questions of individual students and groups of students
- providing extension activities for selected students
- giving particular students opportunities to do different things with the activities



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Chapter 2

Operations

Understand the meaning, use and connections between addition, multiplication, subtraction and division.

Overall Description

Students understand the meaning of addition, subtraction, multiplication and division, as distinct from how to carry out the calculations associated with them. They decide which operation is needed, including in contexts where no obvious verbal cues indicate which operation is expected. For example, they can see "take away" and "comparison" situations as each involving subtraction and can write the appropriate subtraction.

Students recognize the need to multiply and divide in situations involving repeated addition, arrays, rates and conversions, areas, and enlargements and reductions. They also know that multiplication does not always "make bigger" and can say under what circumstances it does and does not. They also recognize and can deal with both familiar and unfamiliar situations involving ratio and proportion.

Students understand key relationships within and between the four basic operations. They use these to construct equivalent expressions, to find unknown quantities and to assist computation. For example, they think of nine as composed of four and five so that $9 = 4 + 5$, but also $9 - 4 = 5$ and $9 - 5 = 4$. Understanding such relationships enables students readily to solve problems such as $\square - 7 = 11$ or even $\square - 348 = 434$.

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BACKGROUND NOTES

Representing Word Problems in Different Ways

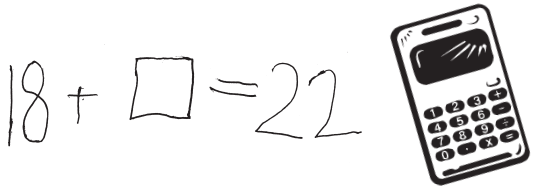
Students need to develop a deep understanding of the meaning and use of the four basic operations—of their links to each other and to real-world applications. In order to build up conceptual links between the various types of situations and the addition, subtraction, multiplication and division operations, a rich and flexible variety of representations is needed over an extensive period of time. It should not be rushed.

These various forms of representation include[1]:

- experience-based scripts of real world events or dramatic play
- manipulatives
- pictures and diagrams
- spoken language
- written symbols in number sentences.

Thus, students should describe, represent and explore both additive (addition and subtraction) and multiplicative (multiplication and division) situations dramatically, physically, diagrammatically, verbally and symbolically.

Many teachers have found a "think board" helps students to link various ways of representing the operations. There are many different versions of, and ways to use a think board. With the simple versions, one problem is dealt with at a time.

Story	**Materials**
If one penguin ate 18 fish and another ate 22, how many more fish did the second penguin eat?	
Picture/Diagram	**Number Sentence**
	

[1] This list is adapted from Verschaffel, L. & De Corte E. 1996, 'Number and arithmetic', in A. Bishop, et al (eds) *International Handbook of Mathematics Education*, Kluwer Academic Publishers, Netherlands, pp99–137.

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Students in the early years might work with a think board similar to the one previous[2], or a simplified version of it. They work in small groups around the different sections of the think board. Teachers might start them with a story and the students then decide which section to work in next. They might decide to start with the materials section and as a group decide to get out blocks to represent the penguins in the previous story, manipulating them until they find the answer.

The students then choose to move to the pictures section and as a group, decide to draw a picture of the 18 fish and the 22 fish. Many students will choose to use a diagram instead of drawing something that looks like the real thing. This is appropriate as children need to move to more abstract representations.

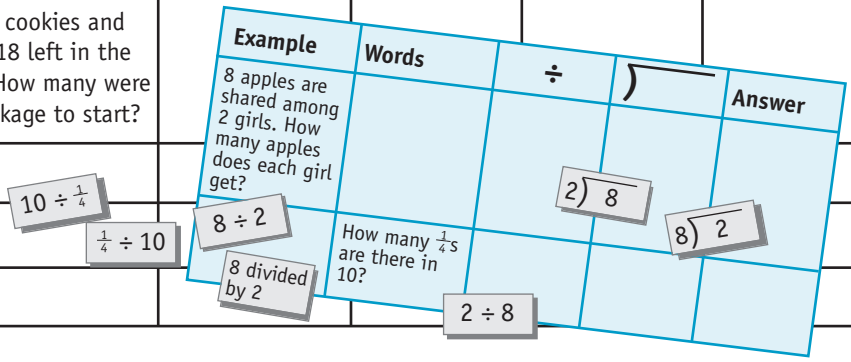
They then work together to decide what number sentence they could write that connects to the way that they have solved the problem. For the previous story, this might be 22 – 18 = ☐ or 18 + ☐ = 22, depending on how they solved it. This section might include a calculator for the students to use. In this case, the focus of this section is on deciding which buttons they need to push in order to solve the problem, that is, the operation.

Students in the later years might work with more complex think boards like the division think board below[3]. In this version, students may be provided with cards that they place in the appropriate squares. They can then work in small groups to discuss and debate their views on the appropriate placement of cards.

The focus of the think board is on finding the connections between the different ways of representing the problem. This helps students to focus on the meaning of the operation rather than on just calculating to find an answer.

The think board

Story	Dramatization (Story picture)	Things	Diagram	Number sentence	Answer
We ate 16 cookies and there are 18 left in the package. How many were in the package to start?					



[2] Haylock, D. W. 1984, 'A mathematical think board', *Mathematics Teaching*, 108, pp.4–5.
[3] Swan, M. 1990, 'Becoming numerate: developing conceptual structures', in *Being Numerate: What Counts?*, Sue Willis (ed), ACER, Hawthorn, Victoria.

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Addition and Subtraction Problems

Students need experience with all the common types of addition and subtraction problems. They *do not* need to learn to name different problem types—the vocabulary of "change", "combine" and "compare" is used here simply to help you ensure that the full variety of addition and subtraction situations are provided. However, this is just one of many possible classifications. Sometimes the same problem can be thought of in different ways and may not obviously belong to one type. The examples below involve small whole numbers and discrete quantities that can be counted. However, students should experience all problem types in contexts involving larger whole numbers and measured quantities, including with decimals and fractions.

Change

where students have to transform one quantity by adding to or taking from it

Join

- Anna had 7 bears and then her brother gave her 3. How many does she now have? (result unknown)
- Anna had 7 bears but would like to have 10. How many more does she need to get? (change unknown)
- Anna had some bears and then her brother gave her 3. Now she has 10. How many did she have to start with? (start unknown)

Separate

- Anna had 7 bears and then she gave her brother 3. How many does she now have? (result unknown)
- Anna had 10 bears and then she gave her brother some. She now has 7. How many did she give her brother? (change unknown)
- Anna had some bears and gave her brother 3 of them. Now she has 7 left. How many did she have to start with? (start unknown)

Combine

where students have to consider two static quantities either separately or combined

- Anna has 7 brown bears and 3 white bears. How many does she have in all? (whole unknown)
- Anna has 10 bears. 7 are brown and the rest are white. How many are white? (one part unknown)

Equalize or Compare

where students compare or equalize two quantities

Students tend to find EQUALIZE problems easier than COMPARE problems, because they suggest an action that students can carry out or imagine carrying out.

- Anna has 10 brown bears and 7 white bears.
  If all the white bears take a brown bear as a partner, how many brown bears won't get a partner?
- Anna has 7 white bears and some brown bears. All the white bears took a brown bear as a partner, and there were 5 brown bears left without a partner. How many brown bears does she have?
- Anna has 10 white bears and some brown bears.
  All the white bears tried to take a partner but there were 4 brown bears too few. How many brown bears does Anna have?

Students tend to find the matching COMPARE problems harder than EQUALIZE problems, because they are static and so do not immediately suggest an action.

- Anna has 10 brown bears and 7 white bears. How many more brown bears does she have?
- Anna has 7 white bears and some brown bears. She has 5 more brown bears than white bears. How many brown bears does she have?
- Anna has 10 white bears and some brown bears. She has 4 fewer brown bears. How many brown bears does Anna have?

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Multiplication and Division Problems

Students need experience in modelling all the common types of multiplication and division problems. The naming of types is used here to help teachers ensure that the full variety of situations are provided for students. It is **not** intended that students are taught to name different problem types.

The two divisions for each multiplication are of different types

Multiplication	Division (partition/sharing) —know how many portions	Division (quotition/grouping) —know the size of the portions
Repeat Equal Quantities		
■ There are five tables and six students can sit at each table. How many students can we seat?	■ Thirty students need to form five equal groups, one at each table. How many students should be in each group?	■ There are 30 students in our class. Our new hexagon tables each seat six students. How many tables will we fill?
■ Nine flags will each need two-thirds of a metre of fabric. How much material do we need?	■ We have six metres of fabric to make nine flags. How much fabric can we use for each flag?	■ Each flag needs two-thirds of a metre of fabric. How many flags can we cut from six metres?
Use Rates		
■ Peaches cost $4.90 a kilogram. How much will it cost for 2.5 kilograms?	■ If a 2.5-kilogram basket of peaches costs $12.25, what is the price for one kilogram?	■ Peaches cost $4.90 a kilogram. If a basket of peaches costs $12.25, how much must it weigh?
Make Ratio Comparisons or Changes (scale)		
■ Tanya has five times as many marbles as Jill. If Jill has 14, how many does Tanya have?	■ Tanya has five times as many marbles as Jill. If Tanya has 70, how many does Jill have?	■ Jill has 14 marbles. How many times as many marbles does Tanya have if she has 70?
■ A picture is 60 cm high. If it is reduced to 0.8 of its original size, what will its height be?	■ A picture that has been reduced to 0.8 of its original size is now 48 cm high. What was its original height?	■ A 60 cm high picture is reduced to 48 cm. What is the reduction ratio?

The two divisions for each multiplication are not of different types

Make Arrays and Combinations

- There are three different types of ice-cream cones and six ice-cream flavours. How many different types of (single) ice-creams can we make?
- There are several different types of ice-cream cones and six ice-cream flavours. If 18 different (single) ice-creams can be made, how many different types of cones must there be?
- There are three types of ice-cream cones and various ice-cream flavours. If 18 different (single) ice-creams can be made, how many different flavours are there?

Need Products of Measures

- What is the area of a rectangle 16 cm by 13.6 cm?
- A rectangle of area 217.6 sq centimetres has one side 16 cm long. How long is the adjacent side?
- A rectangle of area 217.6 sq centimetres has one side 13.6 cm long. How long is the adjacent side?

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Representing Word Problems in Number Sentences

Initially students will solve problems by "playing out" or modelling them in various ways—dramatically, physically, mentally and with drawings—and then counting. However, they do need to learn to represent problems in number sentences containing the operation symbols. Being able to represent problems symbolically helps students to see the connection between apparently quite different problems that can be solved with the same mathematics.

Students need to learn to write number sentences for simple word problems.

An important aim is to teach students to represent problems mathematically. One of the earliest forms of this is to write number sentences to represent simple addition and subtraction word problems.

Most students readily learn to write addition and subtraction number sentences for simple Join and Separate problems where the result is unknown. However, they often have difficulty writing standard number sentences ($3 + 5 = \square$; $8 - 4 = \square$) for other addition and subtraction problem types, even when they can solve the problems by modelling and counting. Researchers[4] have found that even after months of teaching, fewer than half of Kindergarten and grade 1 students could represent a simple Compare problem in the standard form, even though almost all could solve it. They did not connect the sentence writing and the problem solving. The difficulty is that the standard representations often do not match the modelling or counting processes they use.

What these researchers and teachers have also found is that young students can learn to represent the problems if they are taught to use representations, such as $3 + \square = 5$; $\square - 7 = 4$. These are not standard but they more closely fit the meaning of a problem and the way students think about it. Once young students are familiar with this form of writing sentences, they are able to represent the structure of new problems where they have not been taught that particular problem type. That is, they can learn to write number sentences where the representations match *their* understanding of the meaning of the problem.

Students should first learn to write number sentences that match the semantic structure of the problem (its meaning), even though this may produce a non-standard number sentence.

This is surprising at first because we think of standard number sentences as being easier to solve than non-standard number sentences,

[4] Carpenter, T. P. 1985, 'Learning to add and subtract: An exercise in problem solving' in Silver, E. (ed) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*, Lawrence Erlbaum Associates, Hillsdale, New Jersey.

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which first have to be rearranged. But if the problem is not standard, then in order to write the standard number sentence, students have to rearrange the problem in their head, which is much harder. Thus, given a problem like this:

We saw eight butterflies in the garden, but some flew away. Now we only have five left. How many escaped?

students who think of it as *eight and some flew away leaving five* will find it easy to represent it as: 8 – ☐ = 5. It is much harder for young students to think of the above problem as the standard subtraction: 8 – 5 = ☐. If a student cannot solve 8 – ☐ = 5, it is unlikely that he or she will have been able to rearrange the problem in his or her head in order to get 8 – 5 = ☐.

For simple word problems, students do not write a number sentence in one phase and then solve it in a second phase. Much of the solution process is carried out on the semantic representation, not on the mathematical and physical representation of it. That is, students use their understanding of the situation itself to transform the problem before they represent it mathematically—they rearrange the problem in their heads. *After* students have solved a problem, they may write a number sentence that reflects their understanding of what happened in the situation. For the butterfly problem, many will write 8 – 3 = 5 and say, *three butterflies flew away and there were five left*. In order to write the 8 – 3, they need to have *already* solved the problem.

We do eventually want students to be able to write number sentences before they have solved the problem. This will help them solve problems where the numbers are not easily represented in materials or diagrams and where they cannot rely on known number combinations. However, expecting students to begin with the standard subtraction sentence 8 – 5 actually makes it harder. Students should learn to write sentences like 8 – ☐ = 5 and, over time, develop their part-part-whole understanding to link equivalent addition and subtraction sentences, such as 8 = 3 + 5, 8 – 3 = 5, 8 – 5 = 3. We need to help them talk about these links so that they can move between the various ways of representing the situation.

Students should imagine the situation and try to find a helpful way to think about it.

Students should be encouraged to think flexibly about problems and helped to see that we can often think about the same situation in different ways and so represent it differently. For example:

We ate 16 crackers between us and there are 18 left in the box. How many were in the box to start with?

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could be thought of as:

The number in the box take away 16 leaves 18. What is the number?
$\square - 16 = 18$

(A change problem with the start unknown)

or as:

The 18 left together with the 16 eaten gives the number in the box.
$18 + 16 = \square$

(A combine problem with whole unknown)

The two number sentences are equivalent because they represent the same problem; they ask essentially the same question.

Students should paraphrase the problem in a way that makes sense to them.

Many textbooks encourage students to try to represent problems using a syntactic translation of words into symbols. This is unfortunate. It causes many problems and leads to many errors. Good problem-solvers use the semantic structure of the problem, not a syntactic translation. Thus, the process described above of paraphrasing the problem to "see what it is saying" is an essential part of good problem solving and students should be encouraged to do this rather than focusing on key words or phrases.

Students need to link the number sentences from the various ways of thinking about a problem. We need not necessarily go back to the original problem to link them. If we begin with a number sentence such as $\square - 16 = 18$, we can use the inverse relationship that exists between addition and subtraction to rewrite it as $\square = 18 + 16$. The transformed number sentence is easier to solve than the original.

If the student has thought of the *actions* in the cracker problem in the first way described above, then the number sentence $\square - 16 = 18$ is a direct representation of the actions, but the transformed version is not. As was suggested earlier, working with a number sentence that is not a direct representation of the *actions* in the problem is usually more difficult, even though the computational demands may be easier. Unless students believe that such transformed number sentences are always "asking the same question", they will not be able to use strategies flexibly to find unknown quantities and will be forced to use trial-and-error or rote procedures.

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Students should learn to transform problems to assist in problem solving.

In developing skill in solving addition and subtraction problems, students pass through several stages. The less skilled or younger problem-solvers are limited to directly representing the problems. Older or more skilled problem-solvers conduct a more elaborate semantic analysis of the problem and often transform to a form that is easier to solve before they represent it mathematically. Students' solutions to multiplication and division problems follow much the same pattern, although generally not as early.

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Operations:
Key Understandings Overview

Teachers will need to plan learning experiences that include and develop the following Key Understandings (KU). These Key Understandings underpin achievement of this family of concepts. The learning experiences should connect to students' current knowledge and understandings rather than to their grade level.

Key Understanding		Description
KU1	Adding and subtracting numbers are useful when we: ■ change a quantity by adding more or taking some away ■ think of a quantity as combined of parts ■ equalize or compare two quantities	page 32
KU2	Partitioning numbers into part-part-whole helps us relate addition and subtraction and understand their properties.	page 40
KU3	Multiplying numbers is useful when we: ■ repeat equal quantities ■ use rates ■ make ratio comparisons or changes (scales) ■ make arrays and combinations ■ need products of measures	page 48
KU4	Dividing numbers is useful when we: ■ share or group a quantity into a given number of portions ■ share or group a quantity into portions of a given size ■ need the inverse of multiplication	page 60
KU5	Repeating equal quantities and partitioning a quantity into equal parts helps us relate multiplication and division and understand their properties.	page 72
KU6	The same operation can be said and written in different ways.	page 82
KU7	Properties of operations and relationships between them can help us to decide whether number sentences are true.	page 86
KU8	Thinking of a problem as a number sentence often helps us solve it. Sometimes we need to rewrite the number sentence in a different but equivalent way.	page 94
KU9	We make assumptions when using operations. We should check that the assumptions make sense for the problem.	page 102

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Grade Levels—Degree of Emphasis			Sample Learning Activities
K-3	3-5	5-8	
★ ★ ★	★ ★ ★	★	K-Grade 3, page 34 Grades 3-5, page 36 Grades 5-8, page 38
★ ★ ★	★ ★ ★	★ ★	K-Grade 3, page 42 Grades 3-5, page 44 Grades 5-8, page 46
★ ★	★ ★ ★	★ ★ ★	K-Grade 3, page 50 Grades 3-5, page 52 Grades 5-8, page 54
★ ★	★ ★ ★	★ ★ ★	K-Grade 3, page 62 Grades 3-5, page 64 Grades 5-8, page 66
★ ★	★ ★ ★	★ ★ ★	K-Grade 3, page 74 Grades 3-5, page 76 Grades 5-8, page 78
★ ★	★ ★	★ ★	K-Grade 3, page 83 Grades 3-5, page 84 Grades 5-8, page 85
★	★ ★	★ ★ ★	K-Grade 3, page 88 Grades 3-5, page 89 Grades 5-8, page 91
★	★ ★	★ ★ ★	K-Grade 3, page 96 Grades 3-5, page 97 Grades 5-8, page 99
★	★ ★	★ ★	K-Grade 3, page 104 Grades 3-5, page 105 Grades 5-8, page 106

Key

★ ★ ★ **Major Focus**
The development of this Key Understanding is a major focus of planned activities.

★ ★ **Important Focus**
The development of this Key Understanding is an important focus of planned activities.

★ **Introduction, Consolidation or Extension**
Some activities may be planned to introduce this Key Understanding, to consolidate it, or to extend its application. The idea may also arise incidentally in conversations and routines that occur in the classroom.

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Key Understanding 1

Adding and subtracting numbers are useful when we:

- change a quantity by adding more or taking some away
- think of a quantity as combined of parts
- equalize or compare two quantities

This Key Understanding is about the meaning of the addition and subtraction operations and when to use them, rather than how to carry out calculations. Students should learn to recognize a wide range of problem types to which addition and subtraction apply. These should include change situations (add some or take some away), combine situations, and compare and equalize situations. Examples of each of these problem types are provided in the Background Notes on page 24. Students should be helped to see how these types of problems can all be thought of in terms of part-part-whole, and can be solved using the same operations.

Students will gain full command of some problem types earlier than others. However, this does not mean that they should only deal with one type at a time. A wide range of addition and subtraction problems should be posed from the early school years, although the numbers involved may be quite small to begin with. Initially, students should solve problems by acting them out, modelling them with materials and diagrams, and imagining them in their "mind's eye". Students who have reached the end of the Matching phase can do this for self-generated or orally presented stories involving small, easily visualized numbers.

Many students have always come to school recognizing and naming numbers, even if they do not fully understand them. Nowadays they will often meet the operations symbols simply by playing with calculators. They should be provided with a calculator from the earliest years, Kindergarten to grade 3, and encouraged to explore its functions. When they do so, they will want to name various operation keys. Rather than meeting the concepts and attaching the symbols to the concepts later, they are more likely to meet the symbols and, over time, develop and enrich the meanings they attach to them.

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The goal is for students to build connections between dramatic, physical, diagrammatic and verbal forms of problems and the symbolic representations of them.

Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ represent problems involving whole numbers in a variety of ways ■ can link various problem situations to the addition and subtraction operations and the symbols ■ link the various addition situations to the part-part-whole notion, so they understand *why* the addition symbol works in each case ■ link various subtraction situations to part-part-whole and to the subtraction symbol ■ write suitable number sentences
Partitioning	■ deal with all the problem types in contexts involving large whole numbers
Factoring	■ deal with all the problem types in contexts involving large whole numbers where there are no verbal cues to suggest which operation is required, where intuition about the size of the answer may not help to choose an operation, and where measured quantities including decimals and fractions are involved

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Sample Learning Activities

K-Grade 3: ★ ★ ★ Major Focus

Role Play

Have students act out the characters in a story. Each time characters come or go, a child records the number and the operation/action or sign on a board, such as +2, -1. At the end of the story, ask: Which part of the story was the "+2" for? Could it have been when the wolf ran away? Why? Which part of the story could "3 + 1" be about?

Plus or Minus

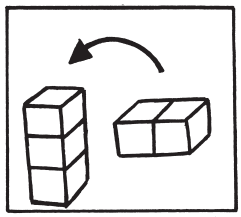
Retell a favourite story to students and have them hold up a card showing either "+" or "–" each time a character joins or leaves the scene. Ask students to compare and explain their choice. Have them write a number sentence for a part of the story.

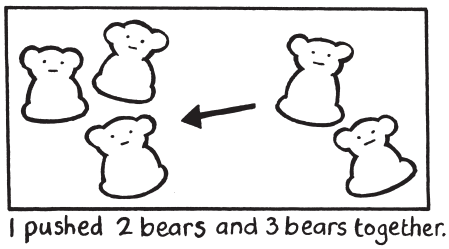
Equal Groups

Have students suggest ways to make equal groups in everyday tasks. As they pack equipment away or distribute materials, for example, ask: What could you do if there should be six glue sticks but there are only two? What could you do if there are eight pencils but there should be six?

Messages

Organize students into pairs. Provide each student with an identical group of objects, such as linking cubes, on each side of a barrier. Have partners take turns to combine or separate their group of objects and show what they did by a drawing or by writing numbers and signs on a message.





Then have them pass the message over the barrier for their partner to replicate the action. The first student then checks to see if it is correct. After several turns, with students swapping the tasks, have them sort and display the messages in "combine" and "separate" groups. Introduce the "+" and "–" signs, or use messages that show numbers and signs as a way to represent the materials and actions. Draw out that different situations can be shown as 2 + 3: I pushed two bears and three bears together; I added two cubes on to my stack of three cubes. (See Case Study 1, page 56.)

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Think Board

Encourage students to show they have made the connections between symbols and real-life situations using a simplified think board (See Appendix: Line Master 1). Ask them to tell a story that is represented by symbols such as 4 + 9 or 13 – 6, and show how this is represented by materials or a diagram.

Separating Objects

Ask students to describe the different ways a group of objects can be separated. Use toy animals, counters or playdough to represent a story, such as: Five birds in a tree. Three flew away. How many birds left in the tree? Focus on the use of terms, such as "flew away," "left" and "three from five." Ask: Which key on our calculator takes away?

Matching Collections

Following *Separating Objects*, above, ask students what they did to match the actual objects to the number of birds in the story. They might say: *I put out five counters to show the birds. Then I took away three counters to show the birds that flew away*. Ask: So what is the difference between five birds that were in the tree and the birds that are there now? Use terms such as "how many more" and "the difference". Record their explanation on the board as they retell it. Focus on their decisions to "add more" or "take away some".

Number Line

Have students use a number line to compare two different-sized groups and say how many more are in one group. Focus on the smaller number and ask: What would you do to the smaller number to make it the same as the bigger number? What would you do to the larger number to make it the same as the smaller number? Which movement shows joining, and which shows separating?

Early Bird

Ask students to describe how they show or think about the amounts for comparison problems. Use a picture of 13 birds and seven worms and ask: Suppose the birds all try to get a worm. Will every bird get one? How many birds won't get a worm? How do you know? How else could we work it out? Ask students to draw the solution and say what they did. Help them write a number sentence for their drawing.

Word Problems

Write different types of change situation word problems based on the same story idea. For example: A farmer had 11 chickens, he bought seven more and then he had 18 altogether; or, the farmer had 18 chickens, seven escaped and then he had 11 left. Present the problems with start, change and result unknown. Have students describe, draw and write number sequences to show how they joined and separated the groups to find the solutions. (See Background Notes on page 24 for further examples of change problems.)

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Sample Learning Activities

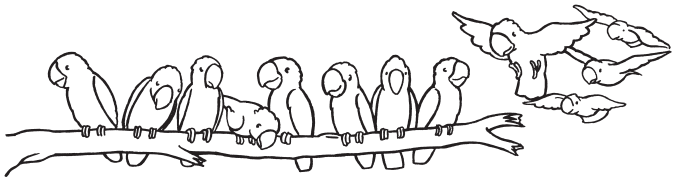
Grades 3-5: ★ ★ ★ Major Focus

Think Board

Draw up a think board (See Background Notes, page 22, and Appendix: Line Master 1). Put an example in one of the sections and ask students in groups to represent the problem in the other sections. For example, start with a story such as: Jonathan has collected 12 toys. Eight are cars. The rest are trucks. How many are trucks? Have students show it in materials, as a picture, as a number sentence, and write the answer. Ask: Which sign did you choose, and why? What was it about the story that told you it was an addition/subtraction?

Story Problems

Give students two story problems using the same numbers and operation: one showing a change situation and the other a combine situation. For example, a change situation: There are eight gray parrots in a tree. Some more birds came. Now there are 25 birds in the tree. How many birds came? Or, a combine situation: There are 25 birds in a tree. Eight are gray parrots and the rest are cockatoos. How many are cockatoos? Ask: If the numbers are the same, what is the difference in these problems? What made you think of them both as addition/subtraction?



Classroom Problems

Have students mentally solve change problems that arise in the classroom. For example: 14 students go home for lunch; how many eat lunch at school? Then have students work out a number sentence to represent what they did. Ask: What made you think it should be an addition/subtraction sentence?

Comparing in the Classroom

Broaden *Classroom Problems*, above, to include combine, and then compare and equalize situations. For example:

- In October and November, 98 mm of rain fell in Vancouver. If 26 mm fell in October, how much rain fell in November? (Combine)
- Simon's sunflower measured 117 mm and Kadan's measured 145 mm. How much taller is Kadan's plant? (Compare)
- There are 114 children going to the barbecue, and each child needs to have a plate. I already have 75 plates. How many more plates do I need? (Equalize)

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Analyzing Word Problems

Have students work out which numbers in a word problem show the whole and which show the parts. They then say whether it is a part or the whole that is not known. For example:

- If two penguins ate 32 fish between them and one penguin ate 19, how many fish did the other penguin eat? If 32 is the whole and 19 is the part, what do we do to find the other part?



Number Lines

Extend the approach in *Analyzing Word Problems*, above. Ask students to solve change, combine, compare and equalize, and addition and subtraction problems using a number line to represent the parts and the whole. For example: A new computer game costs $76. Jo has saved $44. How much more does she have to save?

Writing Problems

Ask students to write problems for others to solve while studying Social Studies or Science and Technology. For example: If Alexander Graham Bell invented the telephone in 1876, how many years ago was it invented? Students need to say which operation is required and why, then decide on the number sentence to use to solve the problem. Ask: How did you know what order to put the numbers in?

Classifying Problems

Ask students to classify word problems according to whether they can be solved using addition or subtraction. Start with sets of problems involving change situations all using the same pair of numbers. Some should include extra numbers. For example: Anna had some hockey cards. Her nine-year-old brother gave her three more. Now she has ten hockey cards. How many did she have to start with? Students will discover they need a third group—problems that can be solved either with addition or subtraction. Discuss the groups. Ask: What do all of the problems in the addition group have in common? What makes them all addition? Note that the "missing number" additions can also be solved with subtraction. (See *Classifying Problems*, page 46.)

Story Problems

Ask students to write a story problem for a given number sentence, such as 17 + ☐ = 48 or 48 – 17 = ☐. Ask: What is in the story that shows it matches the number sentence?

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Sample Learning Activities

Grades 5-8: ★ **Introduction, Consolidation or Extension**

Addition and Subtraction

Ask students to use everyday events as a source of addition and subtraction questions or problems. For example: Sam knew the length of a killer whale was about 9 m. He wanted to get an idea of what 9 m looked like. He marked off what he thought was 9 m but, when he measured, he found that his estimate was actually 7.5 m. Sam wanted to know how much short his estimate was. Ask: Is this an addition or subtraction problem? How do you know?

Classifying Problems

Repeat activities such as *Addition and Subtraction*, above, with the think board (see Background Notes, pages 22 and 23.) Have students provide situations for a number sentence such as 4.6 + ☐ = 9.9, where the change, that is, the addend or subtrahend, is unknown. Ask students to share and classify their problems into those where amounts are joined, compared, made equal or thought of as parts of a whole.

Change Problems

Have students represent change problems that include fractions and decimals. Ask them to say how they decided on an operation and a number sentence. For example:

- Gemma had $4\frac{1}{4}$ of licorice twists. After her brother asked her for some, Gemma broke off $\frac{1}{2}$ of one licorice twist and gave it to him. How much licorice does Gemma have left?
- Gemma has $4\frac{1}{4}$ licorice twists. She gave some to her brother. Now she has $3\frac{3}{4}$ left. How much did Gemma give to her brother?
- Gemma had some licorice. She had given her brother $\frac{1}{2}$ a piece and now has $3\frac{3}{4}$ left. How much did she start with?



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Long Jump

Have students decide which operation to use in equalize situations involving decimals. For example:

- Sonya jumps a distance of 3.25 m and Mark jumps 2.38 m. How much longer does Mark's jump have to be to match Sonya's?
- Mark jumps 2.38 m. If he jumps another 0.87 m, he will have jumped the same distance as Sonya. How far did Sonya jump?
- Sonya jumps 3.25 m. If Mark jumps another 0.87 m, his jump will be the same as Sonya's. What was Mark's original jump?

Part-Whole Situations

Ask students to use diagrams, materials (such as cash register tape and string), number lines, words or mental images to represent the parts and the whole in situations, in order to decide on the operation. For example, using the Long Jump activity above, have students paraphrase the problem to see what it is about: *Mark's jump and some distance will equal Sonya's jump*. Students could then draw a diagram to represent Mark's jump as a part and Sonya's jump as the whole. Ask: Is it a part or is it the whole that you don't know? How does knowing the missing part help you decide on an operation and write a number sentence?

Fractions and Decimals

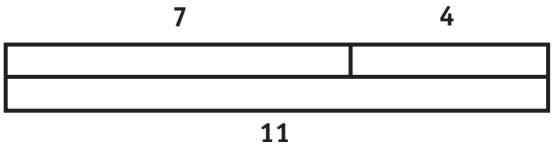
Extend the range of addition and subtraction problems students solve to include fractional and decimal quantities. For example: Yesterday the minimum temperature was 15.2˚C in the morning. By the afternoon, the temperature rose to a maximum of 38.8˚C. How much did the temperature rise?

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Key Understanding 2

Partitioning numbers into part-part-whole helps us relate addition and subtraction and understand their properties.

A quantity, while being thought of as a whole, can also be thought of as composed of parts. That is:



7 + 4 = 11 and 4 + 7 = 11
11 – 4 = 7 and 11 – 7 = 4

The part-part-whole relationship shows how addition and subtraction are related, with subtraction being the inverse of addition. If the whole quantity is unknown, addition is required. If one of the other quantities is unknown, subtraction is required. This enables students to see *why* a problem that they think about as adding—but with one of the addends unknown—could be solved by subtracting. And it enables them to see why a problem that they think about as subtracting—but with the total or starting point unknown—could be solved by addition. Linking the joining and separating of the parts that make the whole to a variety of situations also helps students to see *why* subtraction can be used to solve a take-away problem and also a comparison problem. Understanding part-part-whole relationships helps students represent a problem in different ways, so they can choose the most helpful.

The part-part-whole relationship is also the key to students seeing *why* addition is commutative and *why* subtraction is not. The commutative property of addition is of obvious practical use in calculating, but knowing that, and understanding why, addition is commutative and subtraction is not, helps students represent word problems with appropriate addition and subtraction sentences.

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Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ can solve simple addition and subtraction problems for whole numbers, mostly by modelling strategies ■ may not link addition to subtraction or the types of subtraction to each other
Quantifying	■ link the types of addition to the part-part-whole idea and so understand why the addition symbol works in each case ■ link subtraction types to the part-part-whole idea and to the subtraction symbol ■ can use part-part-whole relationships to link addition to subtraction and so, given 16 + ___ = 34, they could work out a related subtraction and so find the "hidden number" on their calculator
Partitioning	■ use the inverse relationship between addition and subtraction routinely for large whole numbers *For example:* Students may readily say that if 35 + 65 = 100, then 100 – 65 *must be* 35, although they may still rely on imagining it in diagrams.
Factoring	■ can use the inverse relationship in an abstract way for any numbers including decimals and fractions
Operating	■ can use the relationship to solve more abstract "algebraic" problems, such as "If half my number, add one is 43, what is my number?"

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Sample Learning Activities

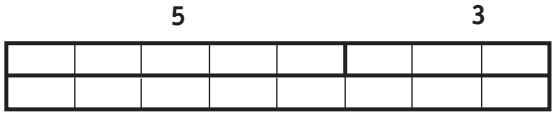
K-Grade 3: ★ ★ ★ Major Focus

Number Combinations

Ask students to represent and record all the combinations of a given number using everyday materials (beans, pop can pull-rings, straws, leaves), story contexts and games. Display the addition number sentences and discuss, focusing on two at a time. For example, ask: Is 4 + 3 the same as 3 + 4? Use your beans to show how it is the same and how it is different. Ask: If Ben has three beans and Fran has four, will there be the same number of beans if they swapped, so that Ben has four and Fran has three? Why?

Part-Part-Whole

Have students use Base Ten Blocks and linking cubes to represent part-part-whole situations. Ask them to write all the number sentences represented. For example:



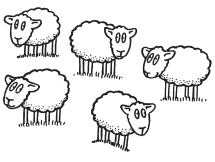
5 + 3 = 8
3 + 5 = 8
8 – 3 = 5
8 – 5 = 3

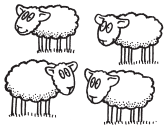
Discuss, focusing students on:

- the two additions
- the two subtractions
- the relationship between the additions and the subtractions

Ten in Bed

Read familiar stories to students, such as *Ten in the Bed* (Dale, 1999). Check addition and subtraction calculations by inverting the problem. Use the characters and events in the story to make up problems. For example: Ten in the bed and one fell out: 10 – 1 = 9. Check this by adding one to nine to see if it is ten. What number sentences could we use if there were ten in the bed and two fell out? What about if three fell out?





Inverse Relationships

Have students use materials (beans, play people, toy animals) and diagrams to model inverse relationships through stories. For example: Have students visualize nine sheep in a paddock. They have to separate them into two groups. Ask: If five sheep move to another paddock, how many will be left here? How do you know? Focus on the idea that knowing how many sheep are in one paddock enables us to know how many are left in the other.

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Think Board

Ask students to use a think board (See Appendix: Line Master 1) to create three different representations of one operation starting from a story (model with materials, draw the situation in a picture, use numbers and symbols). For example: There were 15 bears in the zoo enclosure, but I could only count nine. How many were hiding in their cave? Ask them to compare their representations with those of other students. Ask: What is the same (different) about the things (pictures)? What is the same/different about the number sentences? Can you use both addition and subtraction to show what you did? (See Background Notes, page 22, for an example of a think board.)

Number Sentences

Ask students to write two different number sentences for a combine situation. For example: There are five students in our class with curly hair; how many do not have curly hair? (They could write 5 + ☐ = 27 or 27 – 5 = ☐.) Ask: What is the difference between the two sentences? How does each number sentence tell the story? How would you solve each one?

Number Line

Have students use a number line to solve word problems and then record them in their own way. For example: I had 12 hockey cards and I gave four to my brother. How many cards do I have left? Ask: Can you go both forwards and backwards to solve the same problem and write it as a number sentence? Why? How many cards altogether? Is this the whole number?

Secret Numbers

Show students how to play a calculator game, *Secret Numbers*, to practise inverting number sentences. Organize the students into pairs. One student secretly enters a number into a calculator and then adds a number they both agree on, such as five. The student with the calculator shows their partner what the new number is. The partner says what the original number was and checks it on the calculator by entering a subtraction. For example: The first student secretly enters seven, adds five and gets 12. The partner says "The number was seven", but must check on the calculator by entering a subtraction (12 – 5), not by entering 7 + 5.

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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Related Addition

Ask students to draw pictures showing the whole numbers and part numbers in related addition situations:

- James has 26 marbles and his brother Cameron has 38. Altogether, they have 64 marbles.
- James has 38 marbles and Cameron has 26 marbles. Altogether, they have 64.

Ask: How are these situations the same or different? How are the number sentences the same or different?

Number Relationships

Invite students to use linking cubes, Base Ten Blocks, Cuisenaire rods or a number line to show a relationship, such as 14 + 6 = 20. They can then write as many addition and subtraction sentences as possible. Ask: If I know 14 + 6 = 20, what else do I know? Include examples such as 20 = 6 + 14 and 16 = 20 – 4.

Think Board

Using a more complex think board (See Appendix: Line Master 2) with one box on each line filled, have students determine the other representations for each problem. Ask: Which problems have two ways of writing the number sentences? Why?

Think Board

Story	Dramatization (story picture)	Materials	Picture/Diagram	Number Sentence	Answer
Justin gave 9 stickers to his brother then 16 were left in the package. How many stickers were in the package to start?					
				32 + 17 = ☐	
					

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Addition and Subtraction Links

Focus on the link between addition and subtraction. Pose this problem: Molly and Hasibee were measuring the height of two chickens. One was 29 blocks high and one was 24 blocks high. Molly worked out the difference in the chickens' height by counting from 24 up to 29, and wrote down 24 + 5 = 29. Hasibee counted down five places from 29 until he got to 24 and wrote down 29 – 24 = 5. Ask: Why can we do this problem in two ways. Is one way more correct than the other?

Modelling Problems

Have students use materials (beans, pop can pull-rings, counters) to model problems involving comparison. For example: If one penguin ate 18 fish and another ate 22, how many more fish did the second penguin eat? Ask students to show two ways it could be recorded, using addition and subtraction. Ask: Will either do? Why? Which fits the way you worked it out? Which is easier to work out?

Which Operation?

Ask students to review a collection of stories demonstrating combine, change, compare and equalize situations (See Background Notes, page 24, for examples). Ask them to decide which numbers are being joined and which are being separated, which are the parts and which are the whole. Ask: If the whole is known, which operation can you use to find the missing part? If the whole is unknown, which operation can you use to work it out?

Backwards and Forwards

Following the *Which Operation*? activity above, ask students to represent one of the story problems on a number line. Ask: Do the numbers represent the parts or the whole? Do you locate the numbers directly onto the number line? Does one of the numbers tell you to move a number of spaces along the line? Do you need to move forwards or backwards on the number line to find the solution? Which operation is represented by forwards movements? Which operation is represented by backwards movements? Repeat for other problems. Ask students to write a number sentence for each story.

Unknown Number

Have students decide what number is unknown in a problem: is it one of the parts or the whole? Present a story problem, where one of the parts is unknown. For example: A new computer game costs $76. Jo has saved $44. How much more does she have to save? Ask students to show how it is possible to use both addition and subtraction to solve the problem. Give a problem where the whole is unknown. For example: Jo has saved $44. She received a total of $32 for her birthday. How much does she have altogether? Ask: How can we solve this? Can we use both addition and subtraction? Why? Why not? What is the difference between the first and second problem?

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Classifying Problems

Sort a range of addition or subtraction problems into those that can be solved by addition, subtraction, or either addition or subtraction. Look at problems where one part is unknown. For example:

- Twenty-eight students were in the classroom. Some went outside. If there were 16 left inside, how many went outside?
- Garry Smith drove 133 km on the weekend. He drove 86 km on Saturday. How far did he drive on Sunday?

Ask: Why is it possible to solve each of these with both operations? Which operation is most useful for each problem? Why? (See *Classifying Problems*, page 37). Note that the "missing number" additions can also be solved with subtraction.

Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Long Jump

Have students use part-part-whole models to explain why different operations are equivalent. Ask them to examine their representations and number sentences for the long jump problem (See *Long Jump*, page 39). Ask: How did you use the part-part-whole diagram to arrive at 3.25 m – 2.38 m = ☐? How did you use it to arrive at 2.38 m + ☐ = 3.25 m? Is there another one? Do all these number sentences answer the question? Use your diagram to show why.

Choosing Operations

Have students use what they know about the parts and the whole to practise deciding on operations. Repeat activities where students represent the whole and the parts for a range of addition and subtraction problems using part-parts whole models. For example:

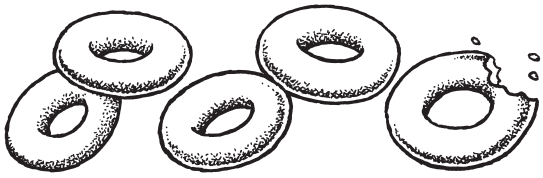
- Sam was saving for an aquarium which cost $176. He was given $85 for his birthday. How much more did he have to save?
- The total snowfall for the three months of winter was 514 mm. In December, 105 mm fell, and 213 mm fell in January. What was February's snowfall?

For each problem ask: What operation is possible if we don't know what the whole is? What operations are possible when either of the parts is unknown? Students use their representations to justify their answers.

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Inverse Relationships

Ask students to use their understanding of inverse relationships to solve equalizing problems. Pose this problem: Sam saw that his brother and friend had eaten most of a box of twelve donuts. There were only four left. Sam said that he should have the rest of the box. Would that be fair? Ask students to represent the problem with materials or diagrams. Invite them to record the number sentences they could use to solve the problem. Talk about the variety of possibilities. Ask: What if there were 5297 donuts left at a coffee shop out of 12 942 that were made for the day? Are the operations the same or different than before? Use your calculator to try to solve this number sentence. What do you find out?



Increased Quantities

Repeat *Inverse Relationships*, above, with other addition and subtraction problems where the quantities are increased. The numbers can also be changed to decimals, including those less than one, and fractions.

Focus on Operations

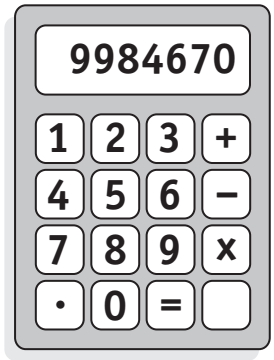
Focus on the relationship between the operations to more easily solve a problem using large numbers. For example: The area of Canada is 9 984 670 square kilometres. The area of the United States is 9 631 418 square kilometres. How much greater is the area of Canada than the area of the United States? Ask students to write and share number sentences to represent the problem:

9 984 670 – ☐ = 9 631 418

9 631 418 + ☐ = 9 984 670

9 984 670 – 9 631 418 = ☐

Ask: Which of these number sentences is useful when using the calculator to solve the problem? Extend to other problems with decimals and fractions.



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Key Understanding 3

Multiplying numbers is useful when we:

- repeat equal quantities
- use rates
- make ratio comparisons or changes (scales)
- make arrays and combinations
- need products of measures

Students should learn to recognize a wide range of problem types to which multiplication applies. They need to be helped to see how these apparently different types of problems are related and can all be solved using multiplication. Examples of the five problem types listed in this Key Understanding are provided in the Background Notes on page 25. Students who understand the meaning of multiplication can also represent number sentences, such as 18 x 3 or 27.6 x 3.2, in materials, a drawing, or a sensible story.

Students are usually introduced to multiplication as "repeated addition". This requires a big shift in thinking from addition or subtraction. To interpret 5 + 2, students can show five blocks and two blocks and then think about what the + means. Interpreting 5 – 2 is a little more complex, but they can show five blocks, then think about what the – and the 2 mean and push two blocks aside. The parts and the whole are visible and the 5 and 2 each refer to a number of blocks. However, for 5 x 2, the 5 and the 2 do not each refer to a number of blocks. One refers to a number of blocks, but the other refers to a number of sets of blocks. If students begin by counting out five blocks and two blocks, they will run into problems when they try to process the x sign. The notion that 5 x 2 refers to five groups of two requires careful development. Students should be helped, from the earliest years, to think multiplicatively about these situations, since repeated addition does not address all situations in which multiplying is helpful.

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Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ can model problems involving "repeating equal quantities" with materials and diagrams *For example:* Give the problem, "To make flowers we need to make five petals for each flower. How many petals do we need for three flowers?" Students are likely to think of it as the addition: 5 + 5 + 5. Given continuous access to calculators, they may think of the × sign in the calculator as a shortcut way of telling the calculator to add three groups of five.
Partitioning	■ can choose multiplication to solve repeated addition problems and also to solve simple familiar rate and combination problems such as, "We have four shirts and three pairs of pants. How many outfits can we make?" These problems do not implicitly involve repeating equal units. However, students' explanations of why multiplication works for rate and combination problems are likely to draw on ideas about repeated addition.
Factoring	■ have begun to understand multiplication as involving more than repeated addition ■ will choose to multiply in situations involving familiar everyday rates (such as shopping) and scales, although these will still tend to involve whole number multipliers or numbers that they think of as "like whole numbers"
Operating	■ have developed multiplicative thinking to the extent that they consistently recognize multiplication as appropriate in situations involving familiar rates, areas, and combinations, including where fractional multipliers are required ■ consistently solve problems of this type regardless of the size of the multiplier, as long as the situation is sufficiently familiar that they recognize that a rate is involved

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Sample Learning Activities

K-Grade 3: ★ ★ Important Focus

Rate Problems

Have students devise simple rate problems, such as: One ant has six legs. How many legs do three ants have? If each child has half a bagel, how many bagels are needed for four people? Ask students to model the situations using materials, such as toothpicks or diagrams, to record six for one, 12 for two, 18 for three, and so on.

Describing

Encourage students to use summaries to describe their groupings from the previous activity as "three sixes", "three groups of/sets of six", and record them as 6 + 6 + 6 and later as 3 x 6. Ask: How many sixes/groups of six are there? How many times did you make a group of six?

Arrays

Have students arrange materials, such as beans and counters, in arrays to find how many will be needed (seats in a bus or seats for a puppet play, seeds in a vegetable garden). Ask: How many in each row? How many rows? Focus students on noticing and describing the array and how many there are altogether. Students can later draw some other arrays and set out materials, use pegboards or draw diagrams and describe different arrays.

Planning

Ask students to collect materials, and focus on equal groups. For example, say: You need three beads each. How many must you get for a group of five people? Ask those who count by ones to try to count by threes. Ask those who skip count how many threes they need. Ask students to write a number sentence and use a calculator to check the result.

Twice as Many

Invite students to use drawings and diagrams to investigate the meaning of "twice as many". Ask: What did you do to the group to show twice as many? Is there a way we could work it out on the calculator? Repeat for "three times as many". Ask: How much bigger is that than twice as many?

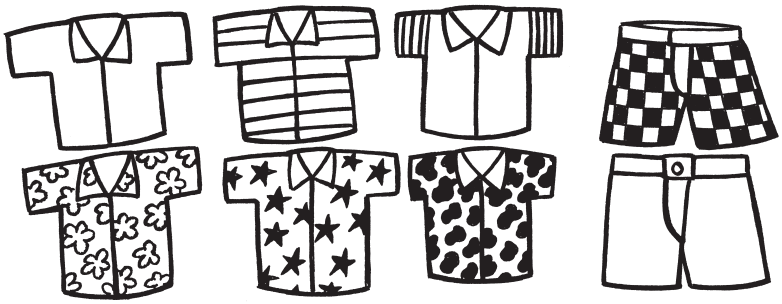
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Five Times as Many

Have students use materials and drawings to model the meaning of "five times as many". Pose the following problem. Sam had three beads and Becky had 12. The teacher said "Becky! You've taken five times as many as Sam." Was the teacher right?

Combinations

Ask students to work out possible combinations where there are two sets of variables. For example: two pairs of shorts and six shirts; six ice-cream flavours and three cones. Students can use models of the items and diagrams to work out a solution. Focus on "how many times?" Ask: How many times did you use chocolate ice-cream?

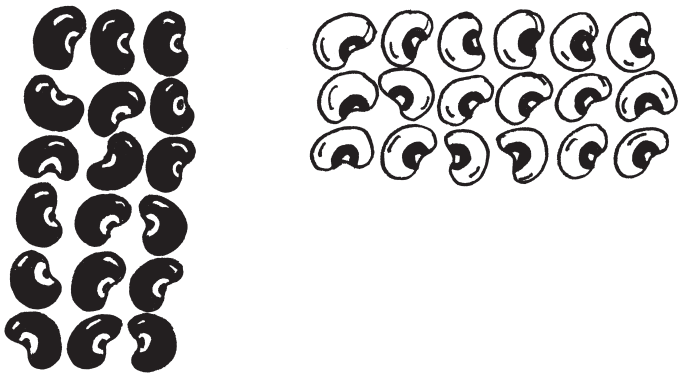


Money

Have students use repeated addition to find the total when buying multiple copies of the same item. Ask them to work out the total cost using money or diagrams. Explain how they know it costs that amount. Students can use their calculator to check each other's totals.

Array Problems

Have students explore arrays to discover that 6 x 3 = 3 x 6, for example. Say: Katy said we planted the peas in rows of three. Tim said they were in rows of six. Could they both be right? Students can use materials, such as beans and/or diagrams, to try to solve the problem.



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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Collections

Have students investigate collections of everyday objects and say how their arrangements help them to see how many there are, such as egg cartons, muffin trays. For example, they might say: *I saw two rows with six in each row.*

Equal Groups

Have students solve problems involving equal groups, such as how many straws will be needed to make five pentagons, five hexagons, and so on. Ask students to write an addition number sentence and a multiplication sentence for each example. Talk about how each sentence shows the same situation.

Grouping and Counting Objects

Have pairs of students take a quantity of blocks. One partner groups or partitions the blocks. Then the other quickly looks at the arrangement and says how many there are: *I knew it was 32 because I saw four in each group and I know 8 x 4 = 32.* Extend the activity to large numbers, using Base Ten Blocks, and fractional numbers, using Pattern Blocks.

Classifying Problems

Classifying Problems on page 37 asks students to classify word problems according to whether they can be solved by addition or subtraction. For example: Anna had some hockey cards. Her nine-year-old brother gave her three more. Now she has ten hockey cards. How many did she have to start with? Pose new problems for the students to classify into those that can and cannot be solved by using multiplication. Ask students to explain their thinking.

Think Board

Provide two multiplication stories, using the same numbers but one involving equal groups and one involving rates. (See Background Notes pages 22–25 and Appendix: Line Master 1.) Ask: How are these the same/different? What was it about the stories that told you they were multiplication problems? What does the x sign mean in each example? What number sentence could be used for both? What does each number in the sentence show? Why are they in that order? What does the x sign show?

Lunch Time

Have students plan a healthy lunch for the class by drawing a picture to work out how much of each item they need to buy and how much it will cost. For example: How much juice will you need to buy if each student drinks 200 mL? If the juice costs $4.70 for 2 litres, how much will it cost altogether? Discuss how the multiplication sign can be used to write a number sentence for each drawing.

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Combination Problems

Have students use materials or diagrams to solve combination problems. For example: How many different types of sandwiches can be made from three types of bread and four fillings? After students complete a number of combination problems involving the same numbers, discuss how the diagrams or arrangements of materials are similar and why multiplication can be used to write a number sentence for each.

Multiplication

Have students decide what the numbers in a multiplication sentence mean in relation to the situation. Use a problem, such as "There were nine people coming to the barbecue and we bought three hot dogs for each", and a number sentence (9 x 3 = 27). Ask: Why can we use 9 x 3 = 27 to represent this problem? How do you know that it is nine groups of three and not three groups of nine? Could the number sentence 3 x 9 = 27 also represent this story? Why?

Rate Problems

Have students write and compare number sentences to solve rate problems. For example: A zoo visit costs $5 per student and $10 per adult, with one adult free for every five students. How much will it cost for 32 students and eight adults? Ask: What number sentences did you choose? Why? What order did you place the numbers in the sentence? Does it matter? What does the x sign show in your number sentence?

Soup for Everyone

Have students solve multiplicative change problems to convert a soup recipe for eight to cater for 16 people then for 12.

Potato Soup (serves 8)
8 small potatoes
1L of water
4 chicken stock cubes
50 ml of butter
250 ml of cream



Ask: What number sentences could be used? In what order should the numbers in the sentence be? Does it matter? Why?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

Combination Problems

Have students solve combination problems. For example: You bought four different coloured shirts and three different coloured pairs of trousers at a sale. Would you be able to wear a different outfit every day for two weeks? How many different coloured shirts would you need to wear a different outfit every day for a month? What would happen if you bought two different hats? How would that increase the number of outfits you could wear? Ask students to write a number sentence in each case.

Comparing Sizes

Have students investigate changes of size. For example:

- Sam uses an overhead projector and enlarges a diagram eight times its original size. The original diagram is 4 cm long. How long will the enlarged image be? How do you know?
- Sam projects a rectangle 13.5 cm wide and 11 cm long onto the chalkboard. The enlarged image is about 44 cm long. How wide will the rectangle appear on the board? How do you know?

Changes in Ratio

Have students use multiplication to solve ratio change problems where the quantity for one measure is not given. For example: Examine a muffin recipe where the quantities given are for 12 muffins. Ask students to adjust the recipe to make six, 18 or 30 muffins. Ask: What would you need to multiply by in each case?

Rate Situations

Have students write number sentences for rate situations using money, decimals and fractions. For example:

- Helen walks at a rate of 1.4 steps per metre. How many steps will she take to walk 100 m?
- When Larissa went to Japan, the exchange rate was 90 Yen to $1 CDN. When Tom went, the exchange rate was 85 Yen to $1 CDN. They both went with $150. How many Yen did they each get?

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Compare Quantities

Have students compare quantities such as distances multiplicatively. For example: Jenny's balloon-powered car went 2.75 m. Shane's went three times that distance. How far did Shane's car travel? June's car went about $\frac{2}{3}$ the distance of Jenny's car. How far did June's car go? How did you work it out?



Calculating Area

Have students identify where multiplication can be used to calculate areas. For example: A painter needs to calculate the area of a wall to know how much paint to order. The wall is 7 m long and 3.5 m high. Ask students to write the number sentence and say why multiplication works in that case.

Decimals

Have students solve multiplication problems involving decimals above and below one. For example: 1 L of unleaded gasoline is 89.9 cents. Ben needs to fill his car. How much will 43 L cost? He also needs to top up his lawn mower. How much will 0.7 L cost? Talk about what operation is required. Ask: Was it the same operation for both situations? Why?

Scales

Have students use multiplication for scales. For example: Tom draws a scale map of the streets around his school. His scale is 10 cm to 1 km. If he lives 2.5 km from school, what should this distance be on his map? His friend lives 0.8 km away. What will this distance be on the map?

Choosing Operations

Have students choose suitable operations for the following problem. Tyler set up a stall selling strawberries from his garden to earn some extra money. He charged $8.50 per kilogram. Circle the operation he would key into his calculator as he weighed each of these amounts of strawberries:

- For 3 kg

  8.50 ÷ 3 | 8.50 – 3 | 8.50 + 3 | 8.50 x 3
- For 3.25 kg

  8.50 + 3.25 | 8.50 x 3.25 | 8.50 – 3.25 | 8.50 ÷ 3.25
- For 1.75 kg

  8.50 ÷ 1.75 | 8.50 + 1.75 | 8.50 – 1.75 | 8.50 x 1.75
- For 0.75 kg

  8.50 + 0.75 | 8.50 – 0.75 | 8.50 ÷ 0.75 | 8.50 x 0.75

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CASE STUDY 1

Sample Learning Activity: K-Grade 3—Messages, page 34

Key Understanding 3: Multiplying numbers is useful when we:

- repeat equal quantities
- use rates
- make ratio comparisons or changes (scales)
- make arrays and combinations
- need products of measures

Working Towards: The end of the Quantifying and Partitioning phases

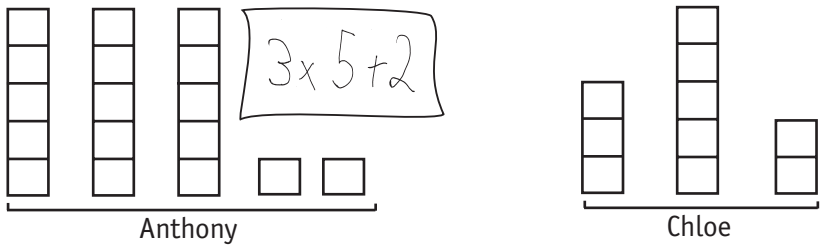
MOTIVATION

Mr. Scott's students were playing a *Messages* game in pairs (See *Messages*, page 34). One student was constructing an arrangement of blocks behind a barrier and writing a summary message (in diagrams or symbols). The partner then was trying to reproduce the arrangement without seeing it. The purpose was to enable students to connect different forms of representation. Inconsistencies between the students' arrangements were providing the conflict needed to motivate them to clarify their thinking about the operations and how they were representing them.



CONNECTION AND CHALLENGE

Mr. Scott thought the students would mostly use addition and subtraction situations, but some produced repeated sets and used a multiplication sign to show their arrangements. For example, Anthony's note read: *3 x 5 + 2*. His partner, Chloe, set out three then five then two blocks, apparently interpreting the x as +. When they compared arrangements, Anthony pointed at the x and told her: *This means three groups of five, not three add five*.



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While only a few students used the multiplication sign, Mr. Scott decided that it was a good opportunity to reinforce what some already knew about multiplication and to develop some preliminary ideas for the others. During the sharing session, Mr. Scott invited students to explain and sort their notes into "Adds" or "Take-aways". He called on Anthony.

Anthony: *Mine's a sort of add but I wrote it the short way. It's a times—3 x 5.*

Chloe: *Yeah, and it tricked me.*

Mr. Scott: *Anthony, which group does your number sentence belong in?*

Anthony: *Well, it needs another group. It's a times.*

Mr. Scott: *Chloe, how was it tricky?*

Chloe: *Well, I thought he put three add five, but it was really three times five.*

Mr. Scott: *What did your blocks look like, Chloe? Draw them on the board.*

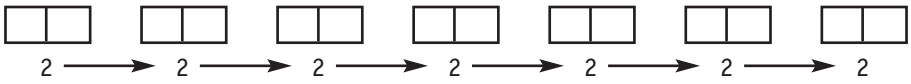
Chloe drew both arrangements of blocks on the board: *Mine was like this, but Anthony had his like this—three groups of five.*

Mr. Scott: *So, how are they different, Chloe?*

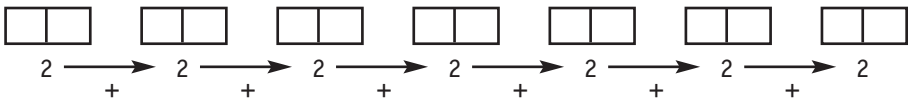
Chloe: *I had ten and Anthony had 17.*

OPPORTUNITY TO LEARN

Mr. Scott asked students to "click" their linking cubes together in twos to make seven little Number Train cars. Then he asked one student to draw this on chart paper, and another to write the quantities for each little car and to show how he would represent this as a message to his partner. Richard drew arrows between the twos to indicate addition:



Mr. Scott asked Marilla if she could write a number sentence for her calculator to say how many cubes. Marilla put "+" signs below Richard's arrows:



Mr. Scott asked the class how many twos Richard had to write down to show all the cubes, and how many times Marilla had to press 2 on her calculator to add all the cubes. In responding to their count, Mr. Scott took the opportunity to model multiplication language: Yes, there are seven twos; seven groups of two; two repeated seven times; so seven times two. Mr. Scott wrote 7 x 2 on the board.

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Anthony explained: *That's a short way of writing adds. Instead of two add two add two add two add two add two, you just write seven times two, and it's like seven groups of two and you don't get tired when you write it.*

DRAWING OUT THE MATHEMATICAL IDEA

To emphasize the multiplicative nature of the process, Mr. Scott wanted the students to focus on the replication rather than simply addition.

Mr. Scott: *What did we do first?*

Students: *We made little cars.*

Mr. Scott: *Yes, we made little cars. When we made a little car, how many cubes did we take from the cube pile?*

Students: *Two cubes.*

Mr. Scott: *So, for each little car you had two cubes. And how many times did we make a little car?*

Students: *Seven times.*

Mr. Scott: *We made the little cars seven times altogether. So, you know each little car is two cubes. How many times did we take two cubes?*

Students: *Seven times.*

Mr. Scott: *Yes, we took two cubes at once, seven times.*

Mr. Scott then moved on to ask the class to think about how many table legs there were in the classroom. Marilla demonstrated her idea by moving around the room touching each table and saying: *Four, four, four ...*

Mr. Scott asked Marilla if she remembered how many she was up to. She stopped, looked back and counted the tables: *That's one four, two fours, three fours ...* and continued on to seven fours. Mr. Scott then went through a similar conversation for the pairs of cubes, emphasizing that the fours were replicated seven times, that there were seven groups of four.

Students recorded 4 + 4 + 4 + 4 + 4 + 4 + 4, and then Mr. Scott asked Chloe if there was a shorter way to write the same thing. Chloe looked confused briefly, but then she wrote 7 x 4: *That means it's there seven times*, she explained. She pointed to the numbers and symbols: *See. Seven times four.*

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Did You Know?

Changing the numbers in problems will often expose underlying gaps or misconceptions in students' understanding. For example, students find the following four problems progressively more difficult, even when they can use a calculator for the actual calculation.

Grapes cost $4.90 a kilogram. How much will 4 kilograms cost?

Grapes cost $4.90 a kilogram. How much will 4.2 kilograms cost?

Grapes cost $4.90 a kilogram. How much will 1.2 kilograms cost?

Grapes cost $4.90 a kilogram. How much will 0.3 kilograms cost?



Thinking about multiplication as repeated addition works for the first problem and students at the Partitioning phase will readily multiply by four. Given a calculator, they may also try multiplying for the second problem, because it has a "feel" of repeated addition. That is, it can be thought of as "four groups and a bit more" and imagined as "skipping along" or adding four groups and a bit more.

However, the same thinking can act as an obstacle to solving the third and fourth problems, and prevent students from seeing multiplication in the situation. Thus, students at the Factoring phase will try multiplication for the second grape problem. Given a familiar context where they have a sense of the sort of answer to expect, they will also solve problems like the third but many will struggle with the fourth. Those students at the Operating phase consistently solve problems of the above type regardless of the size of the multiplier, as long as the situation is sufficiently familiar that they recognize that a rate is involved.

(Note: For diagnostic purposes, the four types need to be presented separately to students, at different times and in different contexts. This way it should be possible to find out whether students recognize multiplication in each situation when there are not other clues to help them.)

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Key Understanding 4

Dividing numbers is useful when we:

- share or group a quantity into a given number of portions
- share or group a quantity into portions of a given size
- need the inverse of multiplication

Students should learn to recognize a wide range of problem types to which division applies. They need to be helped to see how these apparently different types of problems are similar and so can all be solved using division. Examples of these problem types are provided in the Background Notes on page 25. Students who understand the meaning of division can also represent number sentences, such as $6\overline{)24}$ or 7 ÷ 14 or 4.5 ÷ 0.9 in materials, a drawing, or a sensible story.

Students should learn that the division operation is appropriate for problems where you know the quantity and the number of portions to be formed from it, and you want to find how many or how much will be in each portion. For example: Kerri shared 18 cm of ribbon equally among three people. How much did Kerri give each person? These are called partition problems, because you know how many parts. They are also informally called sharing problems.

Students should also learn to use division for problems where you know the quantity and how many or how much is to be in each portion, and you want to find out how many portions there will be. For example: Jason had 18 cm of ribbon and gave each person 3 cm. How many people could get ribbon? These are called quotition problems, because you know the quota. They are also informally called grouping, measuring or repeated subtraction problems.

In the first situation above, the pieces of ribbon are 6 cm long. In the second, they are 3 cm long. The situations look different if represented with materials, but in each case 18 ÷ 3 is the correct calculation. Students should be helped to see how the situations are linked and, hence, *why* division works for each. For example, the sharing problem, "Share 18 jelly beans among three people", can be solved by a grouping strategy. Take out a group of three jelly beans and say "one each", take out another

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group of three and say "two each". Continue until the jelly beans are all gone. Ask: How many groups of three did I take out? That is how many each person gets.

If students associate division mainly with sharing and think of 15 ÷ 5 only as "15 things shared into five groups", they may resist the notion of dividing by fractional amounts. If they associate it mainly with grouping and think of 15 ÷ 5 only as "how many fives in 15", they may resist dividing a smaller number by a larger number. It is important that problem types vary and different language patterns are used from the beginning.

All multiplication problem types have corresponding division problems. The following problems could be set up as a "missing number" multiplication or division.

- If I ask for $40 worth of gas and gas costs 75 cents a litre, how many litres should I get?
- Only 50 people came last week but 125 came this week. How many times as many came this week?

The inverse relationship between division and multiplication should receive careful attention. There are division problems associated with rates, ratio comparisons or changes, arrays and combinations and products of measures. Students need experience with all of them.

Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ use materials and diagrams to represent sharing and grouping situations involving whole numbers and solve simple division problems ■ may base their thinking on counting and/or subtracting equal quantities, and they may not connect the two types of situations
Partitioning	■ have begun to link these types of division where the devisor is the bigger number
Factoring	■ have enlarged their repertoire and apply their knowledge of division to decimals (for money and measurements), although this will still largely be restricted to whole number divisors
Operating	■ have developed flexibility with division, and use its inverse relationship to multiplication for each type of multiplication situation ■ use division in situations where the divisors are decimal and fractional numbers, and may be bigger than the number being divided into

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Sample Learning Activities

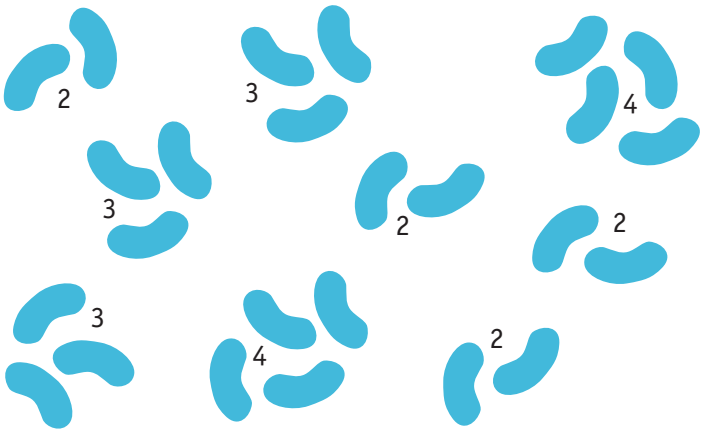
K-Grade 3: ★ ★ Important Focus

Sharing Situations

Have students use materials or diagrams to model sharing situations. For example: There are 24 students for six groups. How many students will be in each group?; Share 15 shapes among five students. Focus on words and strategies such as "dealing" the "same amount each time" to "make equal groups". Ask students to explain how they shared the quantity. Repeat these activities, grouping collections into small equal groups to find how many groups can be made.

Counting a Collection

Have students count a collection of 25 to 35 objects and scatter it. Ask them to then organize and label equal-sized groups, such as threes.



2 groups of 4

3 groups of 3

4 groups of 2

Lead them to summarize the groupings. Ask: How many groups of three/four/one are there? Use this to count again.

Sharing

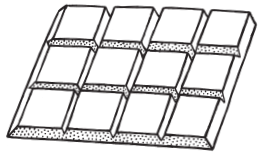
Use diagrams in real sharing situations to work out group sizes. For example: There are 30 students at school today. We need six new groups. How many will be in each group? Help students to use a diagram, such as circles and tally marks to represent the groups and students. Focus on "share out" and "deal out". Ask: What size are the groups? Help students decide on a sensible way to deal with the extras when, for example, only 29 students are at school, and record the result of sharing, say, 29 into six groups.

Grouping

Repeat *Sharing*, above, for grouping. For example: 32 students need to work in groups of six. How many groups will there be?

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Sharing Problems



Use grouping to solve sharing problems (See Appendix: Line Master 4). For example: Share this chocolate (12 pieces in a 3 x 4 array) between three students, with no leftover pieces. How many pieces will each get? Make the link by asking: How many pieces will I break off to give one piece each? If I break off another three pieces, how many will they each have now? How many strips of three can I break off to share it all out? How many pieces will each have altogether?

Describing

Look at these problems: We need three straws to make one triangle. How many triangles can I make from 12 straws?; I can see eight wheels. How many bicycles? Have students use diagrams and tables to find how many groups.

Recording

Have students read and use both symbols for division:

- For $6\overline{)18}$, read: How many sixes in 18?; 18 into sixes; How many sixes make 18?
- For 18 ÷ 6, read: 18 grouped into sixes; 18: How many sixes? Help them to use the division key on the calculator.

Modelling

Have students use materials that closely model sharing and grouping situations, such as pegboards or grid paper, for array word problems. For example:

- When James packed away the 30 shoeboxes, he made each stack five boxes high. How many stacks did he make?
- James made six stacks when he packed away the boxes. How many were in each stack?



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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Think Board

Using a think board (See Appendix: Line Master 1), have students solve problems beginning with a story. For example: 30 students need to form five equal groups, one for each table. How many students should be in each group? Ask: What was it about the story that told you it was a division problem? What does the ÷ sign mean in each example? At other times, begin with a number sentence. Extend into problems involving rates and change. (See Background Notes page 25 for examples of problems.)

Sharing and Grouping

Ask students to use a think board (See Appendix: Line Master 1) to solve the following problems.

- Sharing: 48 horses are placed into six paddocks. Find out how many horses are in each paddock.
- Grouping: Organize 48 horses into paddocks, with six horses per paddock. Find out how many paddocks will be needed.

Ask: How are these two problems the same? Different? What number sentence could be used? Why are the numbers in that order? What does the ÷ sign show?

Calculator

Have students use a calculator to solve sharing and grouping problems (See Background Notes page 25 for examples). After they have done several, ask them to pretend the ÷ key is broken and find another key to use to solve the problems. Ask: How does the symbol you chose relate to the division operation?

Less in a Group

Ask students to use materials to see that when sharing there will be less per group as the number of groups increases. For example:

- Would you rather be in a group of five students who share 30 strawberries, or a group of three students who share 30 strawberries?
- In *The Doorbell Rang* (Hutchins, 1986), look at how the number of cookies changes each time more people arrive and why.

Sharing Strategies

Have students use groups of materials, such as beans and cubes, to solve a sharing problem. For example, to share a raffle prize of $36 among four students, make one group of four (saying "that's $1 each"), make another group of four ("that's $2 each"). Continue grouping in this way. Ask: How much does each student get? Is the answer the number of groups or the number of dollars in each group? Ask students to use a division sentence to represent the original sharing

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problem and then a division number sentence to represent the grouping strategy. Draw out that thinking about the problem as a sharing or a grouping problem will give the same division number sentence and the same result. Then have students use a grouping strategy to solve similar problems.

Array Model

Use an array model to help students see how one problem can be solved by grouping or sharing. For example: There are 24 trees planted in four rows. How many trees in each row? There are 24 trees with six in each row. How many rows of trees? Both problems can be represented by a 4 x 6 array.

Changing Quantities

Have students work out the effect of changing either the total number to be shared, or the number of groups needing a share. For example, after reading *The Doorbell Rang* (Hutchins, 1987), ask students if you would get more cookies if there were:

- 3 students sharing 12 cookies or three students sharing 15 cookies
- 3 students sharing 12 cookies or six students sharing 12 cookies

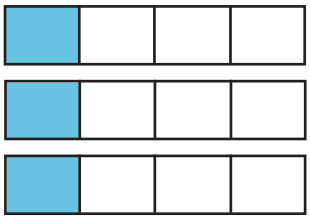
Ask: Which situations will always give more per student? Why?

Relay

Challenge students to investigate a problem where division could help but is not obvious. For example: Find out the distance each member of a relay race would have to run in a 26-km marathon relay. Students choose the size of their relay team but it should have between three and ten runners. (See Case Study 2, page 68.)

Sharing Diagrams

Have students draw a diagram to help solve problems, such as sharing three bars of chocolate among four students. Ask them to write a number sentence to show what they did. Ask: Why is the answer a fraction?



$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$
so $3 \div 4 = \frac{3}{4}$

Different Views

Pose the following problem. James saw $9 \div 3$ and $3 \div 9$ written on the board and said, *That is nine divided by three, and three divided into nine, so they are the same*. Is he right? Carrie said, *You can't do $3 \div 9$ because nine is bigger than three*. Is she right? Sam said, *You can do both but they are not the same*. Ask students to say whom they agree with and to explain their thinking.

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

How Many Each?

Have students group to solve sharing problems. For example, say: Predict what each share will be if 420 carrot sticks are to be shared among 12 students. Ask: How many will I need to pull out for students to get one carrot stick each? What if I pull out another group of 12, how many each? How many have I used up so far? How many groups of 12 will I need to pull out to share all the 420 carrot sticks? How many will each student have? (Note the number of groups gives the number in each share.)

Catering Problem

Extend *How Many Each?* above. Ask students to use the grouping approach to sharing to solve catering problems. For example: There are 125 mini pizzas and 30 students. How many pizzas for each student? If they have one each, that is 30 gone (pull out a group of 30); if they have two each, that is 60 gone (pull out another group of 30) and so on. Ask students why a caterer might prefer to pull out groups of 30 rather than make 30 piles.

Multiplication and Division

Have students use the relationship between multiplication and division to represent sharing problems in symbols. For example: How many pens will each student get if 24 are shared among six students? Ask students to paraphrase to come up with each of the following number sentences: 24 ÷ 6 = ☐ ; ☐ x 6 = 24; 6 x ☐ = 24. Ask: Which number sentence best matches the question? Which number sentence matches the way you might calculate the answer using your tables? Which matches the way you would get your answer using your calculator? Explain how one relates to the other.

Sharing or Grouping

Ask students to investigate division problems to distinguish the actions of sharing and grouping. Include decimal fractions and common fractions and numbers between zero and one. Have a range of division word problems written on cards:

- If 21 litres of juice is bought for a class of 28 students, how much will they get each?
- How many students will get 0.75 litres of juice from a 21-litre container?

Ask pairs of students to classify the cards according to whether they are sharing or grouping situations, and give reasons for their classifications. Have them say the number sentence required to solve the problem.

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Target Division

Play the game in the Did You Know? activity on page 81, varying it by using only the division sign.

Think Board 1

Present a range of problems requiring a number sentence, such as 5 ÷ 8. Ask students to use materials (paper strips, modelling clay, straws) or a diagram to solve the problems and to give their answer as a fraction. For example:

- Jana needs to cut a 5 m piece of wood into eight equal parts. How long will each part be?
- Mario is serving equal portions of five pizzas to eight people. How much will each get?

Ask students to explain how they thought about breaking up the materials. Why is the answer $\frac{5}{8}$ in each case? Is it a coincidence that the numbers in the question are the same as those in the answer? Draw out that five things shared into eight parts is $\frac{5}{8}$ each, 5 ÷ 8 is $\frac{5}{8}$, and $\frac{5}{8}$ means 5 ÷ 8. Ask students to present the link between sharing division and the fraction notation on a think board (See Background Notes, page 22, and Appendix: Line Master 1).

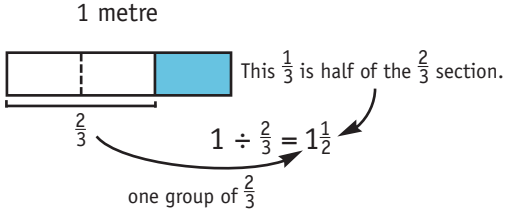
Think Board 2

Have students explore the link between grouping division and the fraction notation on a division think board (See Background Notes, page 23, and Appendix: Line Master 3). Ask them to write sensible problems for:

$1 \div \frac{1}{2}$ $\quad$ $1 \div \frac{3}{4}$

$1 \div \frac{2}{3}$ $\quad$ $1 \div \frac{5}{8}$

Students should then draw diagrams to represent the problems and explain their thinking. For example, for $1 \div \frac{2}{3}$, they may write: *How many $\frac{2}{3}$ m lengths of ribbon are there in 1 m?* and draw:



They explain: *There is one full $\frac{2}{3}$ m length, and a third of a metre left over, but that is half of a $\frac{2}{3}$ m length. So there are one and a half lengths of $\frac{2}{3}$ m.* $1 \div \frac{2}{3} = 1\frac{1}{2}$ or $\frac{3}{2}$. Students investigate other problems and explain why $1 \div \frac{2}{3} = \frac{3}{2}$, $1 \div \frac{3}{4} = \frac{4}{3}$ and $1 \div \frac{5}{8} = \frac{8}{5}$.

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CASE STUDY 2

Sample Learning Activity: Grades 3-5—Relay, page 65

Key Understanding 4: Dividing numbers is useful when we:
- share or group a quantity into a given number of portions
- share or group a quantity into portions of a given size
- need the inverse of multiplication

Key Understanding 9: We make assumptions when using operations. We should check that the assumptions make sense for the problem.

Working Towards: The end of the Partitioning phase

TEACHER'S PURPOSE

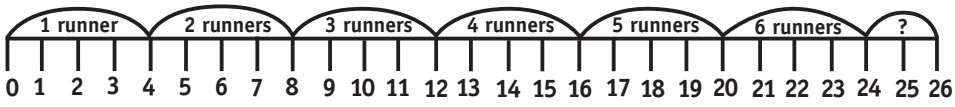
Ms. Ramsey gave her grade 5 class the following task, because she wanted to extend their understanding of division to situations in which division was not explicit.

> *Find out the distance each member of a relay race would have to run in a 26-km marathon relay. You may choose the size of your relay team, but it should be between three and ten runners.*

The task was designed so that:
- whatever the size of the team, there would be a remainder to deal with
- it could be easily completed using division and a calculator, but would be difficult using materials and trial and error
- it had two unknown aspects (the number in each team, the distance each runner would travel), which the students had to deal with

ACTION

Ms. Ramsey was surprised by what students did. Adam decided each runner should travel 4 km. He marked off groups of 4 km on a number line and counted the number of runners, but was then baffled by the extra 2 km.



Shari used a tape measure, pretended centimetres were kilometres and skip-counted in 5 km groups, tallying the number of runners as she went. When she had 1 km left over, she started again with 4 km, then 6 km and 7 km. She finally decided 5 km was closest and asked if she could change the length of her marathon so that five runners ran 5 km each.

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5, 10, 15, 20, 25... 𝍸 5 runners (1 km left)
4, 8, 12, 16, 20, 24... 𝍸 I 6 runners (2 km left)
6, 12, 18, 24... IIII 4 runners (2 km left)
7, 14, 21... III 3 runners (5 km left)

Raymond chose blocks to represent kilometres and attempted to group them so that he could choose a team of runners who would do the same number of kilometres each. He became frustrated after exhausting all the possibilities, saying: *It doesn't work out evenly. There's always leftover bits.*

Ms. Ramsey had assumed students would decide the team size and share the distance among the team. But most chose a whole number of kilometres for each runner and then tried to work out how many runners there would be. They did not think about their assumptions or about what made sense in the situation. Her second surprise was that, although most knew a division procedure with remainders, and all could confidently use a calculator, only one saw right away that division would help.

A few students did take a sharing approach. They folded a 26-cm tape in two, then refolded it to make four or eight equal sections, saying four runners or eight runners would go "that far" each. But even they did not start with the idea of choosing the number of runners. It was more that folding the tape seemed like a good thing to do.

CONNECTION AND CHALLENGE

Ms. Ramsey asked the students to think about what they could do with their "leftover bit". Some thought one runner could run a bit more or less than the others, but Ms. Ramsey explained that in relays everyone had to run the same distance.

She suggested they explain to their partner how many people would be needed if each ran 6 km. Ms. Ramsey asked for comments and the class agreed that four people would be too few but five too many. *Oh well,* Ms. Ramsey said, *We'll make the teams have four and a bit runners!*

This comment got a laugh. When Ms. Ramsey asked what was funny, several students said: *You can't have part people.*

Well, she said, *if we know the number of people has to be a whole number, couldn't we start with that?*

The class thought about what would happen if there were just two runners, and students easily concluded each would run 13 km. At this point, students started to see that splitting the distance equally between the runners was a possibility. *Maybe all the runners could run a tiny bit further to make up the extra*, Casey suggested. Everyone seemed to see the sense of this. To focus their thinking Ms. Ramsey asked them to tell their partner what the class had decided and why, and then to write it down in their own words. Most were able to say that the number of runners had to be a whole number but, to Ms. Ramsey's surprise, still did not recognize that division could help.

Ms. Ramsey asked the class to write it in their own words, because she wanted all the students to think about the assumptions they were making and whether they made sense.

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FURTHER ACTION

Ms. Ramsey asked the students to use lengths of cash register tape marked in 26 units to approximate the distances run for teams of three, four and five runners. As soon as they all had some results, she wrote on the chalkboard a consensus of the distances each runner would have to travel, using students' own language.

3 runners—8 and a half km and a bit more
4 runners—6 and a half km
5 runners—5 km and a tiny bit

To deal with the fact that they did not choose division, Ms. Ramsey wrote her big question on the board:

> *Is there a way to use numbers instead of the tape to share out the kilometres between the runners?*

As the students began working, Ms. Ramsey noticed that many brought out their calculators and a few had actually divided 26 by the team size. She decided not to draw attention to this yet, but asked everyone to use their calculators to help them find a "mathematical instruction" that would work for this.

In this classroom, the students used calculators that dealt with both decimals and fractions. This made this task much simpler than if the calculator only dealt with decimals. However, depending upon their previous experience, students may be able to check the fractions by adding along a number line, or by changing the fraction to a decimal. These strategies may be more suitable for students requiring Grades 5-8 activities.

REFLECTION

After a short time, Ms. Ramsey asked if anyone had anything to share. Sharon volunteered and wrote 4 x 6.5 = 26 on the board. She explained: *It's what my four-runner race is. The 6.5 is six and a half and that's how far the runners did, and I "timesed" it by four and got 26, and 26 is the whole race. But, I don't know how to do it for the other one. I can't put in "five and a bit".*

Kieran was keen to show that 26 ÷ 4 = 6.5 worked better. Ms. Ramsey asked him to explain what he meant.

It's like sharing out, dividing out like 26 between four, that's how many on the team, and then it tells you 6.5 km they go. It's better because you can do it for the other ones, too. When you divide, you don't have to know what that number is [the distance each runner travels]. *The calculator will do it.*

Students should also develop the more general idea that whenever a quantity is divided into equal parts, the number of shares or parts must be a whole number but the size of each share or part need not be.

DRAWING OUT THE MATHEMATICAL IDEA

The class then jointly wrote Kieran's instruction on the board: "Number of km in the race ÷ Number of runners = Distance each runner goes. For this problem: 26 ÷ Number of runners = Distance each runner goes."

Everyone tried it out for their three- and five-runner teams and the class wrote the results in decimals and fractions. Ms. Ramsey then focused the students' attention on whether the numbers made sense. She used their tapes and a fraction number line to help them connect 8.6666 to their earlier approximation of "8 and a half and a bit more", and 5.2 to "5 and a tiny bit".

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The students used division to find the distances for other team sizes and compared their results with approximations using the 26-unit cash register tape. So that the key mathematical points were not lost in all the calculating, Ms. Ramsey gave the students a minute to decide on two or three things that really helped the class solve the problem in the end. Then, she drew out two points through discussion:

- First, thinking about what made sense in the situation was important because it helped us realize that it was the runners rather than the distance that had to be a whole number.
- Second, using division was much easier than trying to do it with tape and other materials.

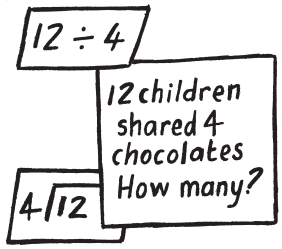
Did You Know?

Students may think that 6 ÷ 18 is the same as 18 ÷ 6, read them as "six into 18" and "18 shared into six" respectively, and think the answer to each is three. The rule they use is "you always divide the bigger by the smaller". Many resist dividing a bigger number into a smaller number. Other students will read division statements in the order in which they come so that they think $6\overline{)18}$ is the same as 6 ÷ 18.

A Diagnostic Activity

Make a set of cards using two different pairs of numbers (say, 5 and 15; 12 and 4) with cards for each of the four possible division sentences for each pair of numbers: 5 ÷ 15, $5\overline{)15}$, $15\overline{)5}$, 15 ÷ 5, 12 ÷ 4, $4\overline{)12}$, $12\overline{)4}$, 4 ÷ 12. Make cards with a variety of word problems involving either 5 and 15 or 4 and 12 (See Background Notes, page 25). Ask students to sort the word problems into piles. Then ask them to put the number sentences that can be used to solve that pile on top of each.

Some students will only have two piles, others four and others eight. If students put all the problems involving the same pair of numbers together, suggest that they think about whether the answer to these problems should be bigger than or smaller than one and sort according to that. Use the conflict this may produce as a basis for discussion. If students correctly sort the problems but label them with the inappropriate number sentences, have groups of students debate what each of the signs mean, how they are entered on a calculator, and so on.



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Key Understanding 5

Repeating equal quantities and partitioning a quantity into equal parts helps us relate multiplication and division and understand their properties.

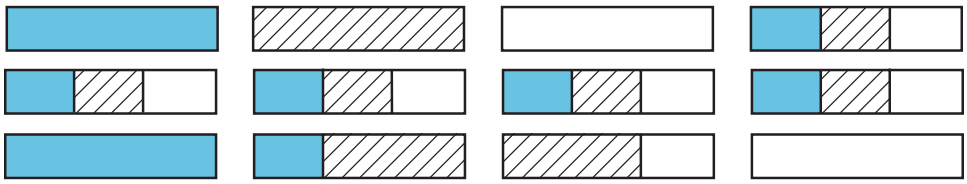
Linking the two ideas of repeating equal quantities and partitioning a quantity into equal portions can help students to understand the connection between multiplication and division. Therefore, it is an important component of their being able to use multiplication and division flexibly to solve problems.

Early understanding of multiplication and division with whole numbers requires students to think about three quantities: the whole (or total) quantity, the number of equal groups, and the amount in each group. If the whole quantity is unknown, multiplication is required. Students have to think about forming equal groups or quantities and then repeating or replicating those quantities a given number of times. If one of the other quantities is unknown, division is required. Students then have to think of taking a whole quantity and partitioning it into a given number of portions or portions of a given size. When we are dealing only with whole numbers, these processes produce the multiples and factors of numbers that are developed in Patterns and Algebra, Key Understanding 6, page 270, and Computations, Key Understanding 3, page 134.

Multiplication is more difficult to conceptualize when the multiplier is not a whole number. Students have to be able to expand their thinking about multiplication from three groups of four, or three bunches of a quantity of four, to include the idea of one-third of a group of four, or one-third of a quantity of four. Even though multiplication is formally involved, the operation becomes essentially one of division. Finding one-third of four is the same as dividing (partitioning) four into three equal portions. In practical terms, sharing four chocolate bars between three people could be done by giving each person one bar, splitting the remaining bar into three parts and giving each person one part. We might give each person one-third of each bar.

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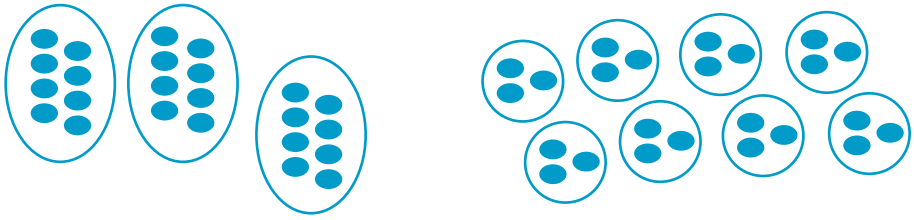
Other practical partitions are also possible:



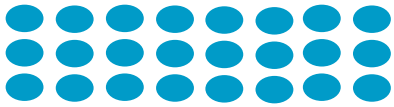
So: $\frac{1}{3}$ x 4 is $\frac{4}{3}$ or $1\frac{1}{3}$

Thus, multiplying by a number that is less than one produces a "smaller" answer. The three ideas of multiplication, division and fractions are closely related.

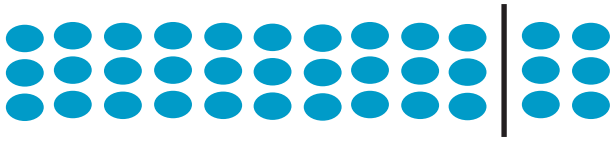
In the following arrangements, the amount in each portion and the number of portions are different, so that two different multiplications are involved—eight taken three times and three taken eight times:



Nevertheless, the products are equal. That is, the operation of multiplication is itself commutative. In concrete terms, three groups of eight is different from eight groups of three, but the total quantity 3 x 8 will always be, *must be*, the same as 8 x 3. An array diagram can help students to see why it must be so:



Arrays can also help students understand the distributive property. Thus, students can see that an array can be split into parts without changing the total:



3 x 12 = (3 x 10) + (3 x 2)

Experimentation and recording alone is unlikely to be enough to develop this Key Understanding. The joint idea of repeating equal groups and partitioning into equal groups, and the associated idea that a number can be decomposed and recomposed into its factors in a number of ways without changing the total quantity, must be drawn from the activities.

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Sample Learning Activities

K-Grade 3: ★ ★ Important Focus

Describing Arrangements

Have students make arrays to show commutativity. Ask students to make cards showing coloured squares, dots, stars or animals arranged in equal rows and columns. They should then describe the arrangements and write a label for each card, such as "Colin has made five rows of three". Ask students to rotate each card and describe the arrangement from that view. Ask: How do you know if there are still the same number?

Describing Quantities

Ask students to group materials, such as beans and cubes, to count more efficiently and describe quantities in different ways. For example, when making arrays, ask students to describe the numbers of rows and columns: *I saw four rows with three in each row. I've used four, three times.*

Composite and Prime Numbers

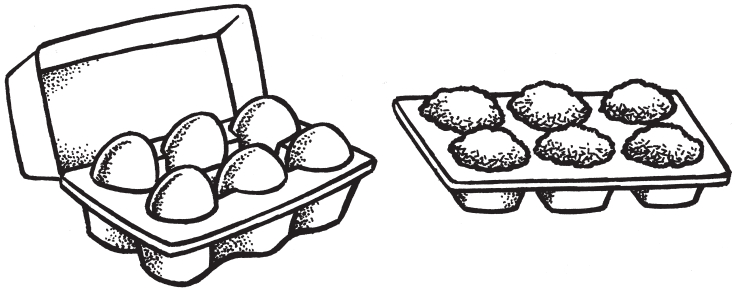
Have students arrange materials, such as beans and cubes, to work out all the possible different equal groups that can be made from a given number. For example: There are 30 students here—could we have equal groups for Show and Tell today? What size groups could there be?

Grouping Summaries

Following *Composite and Prime Numbers*, above, ask students to make diagrammatic displays of the possible groupings if some students are absent. Have students summarize the groupings: *For 29 we can only have a group of 29 or groups of ones. There are no other equal groups in 29. For 27 we can have groups of nines and threes.*

How Many?

Have students investigate collections of everyday objects and say how their arrangements help them to see how many there are, such as egg cartons, muffin trays, packages of gum. For example, they might say, *I saw four rows with three in each row.*



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Halves and Quarters

Investigate practical situations where a smaller number of items is to be shared between a larger number of people. For example: Here are two muffins. If each student gets half a muffin, how many students will this feed? Will there be more pieces or fewer pieces if we cut the two muffins into halves? Ask students to model the problems and explain what they did and found. Repeat for quarters and for other situations. For example: Eight children need to have an equal share of two apples. How much of an apple will each child get?

More or Less

Have students use materials such as playdough to model simple situations that connect multiplication and division. For example: Sven ate four sandwiches. Nikki ate twice as many sandwiches as Sven. Cody ate half as many sandwiches as Sven. Ask: Who ate more and who ate less than Sven? How do you know? What did you have to do to find out?

Retelling

Retell or make up class versions of stories about giants, such as *Jack and the Beanstalk*, adding parts that describe how many steps Jack has to run for each of the giant's steps. Students mark out three of the giant's steps by making one giant step equal to, say, four of their own. Ask: How many steps will you take to show six of the giant's steps (six groups of four steps)? What about half of a giant's step? Have students then represent these, using little cubes for their own feet.

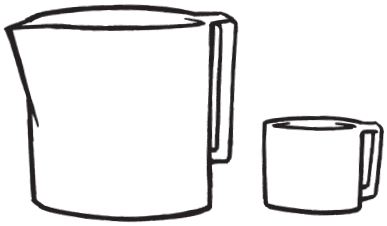
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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

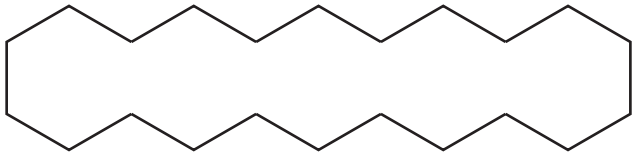
Jugs

Organize students into groups, and have each group work out how many little cups it takes to fill a jug. They then take a number (say, seven) of identical jugs and fill them. Ask: Is it possible to say how many cupfuls that would be? Model the process with students by asking: Do you need to measure the cups in each jug? How many cups are there in each jug (say, four cups)? So there are four cups in each jug, and how many times did we take four cups (seven times)? So we took four cups, seven times; we took seven groups of four cups. Have students draw a picture and record: *Seven groups of four cups. 7 times 4. 7 x 4.*



2-D Shapes

After completing *Jugs*, above, have students say how many hexagons will cover a Pattern Block shape (an outline of a shape produced by tracing around five hexagon shapes placed together), then how many trapezoids fit on the hexagon. Use this to say how many trapezoids will fit on the original shape. Focus on the five times two, rather than counting by 2s. Repeat to find out how many blue rhombuses (five times three) and how many triangles (five times six) will cover the same shape.



More or Less than One

Pose the following problem: Six students divide 18 bars of chocolate between themselves. How much will each one get? More or less than one bar? Do you need to work out how many each gets in order to answer the question? What if some students leave early and do not share the chocolate bars, so there are only four left to share the 18 bars? Will those remaining get more or less now? What if the four students had only six chocolate bars to divide between them? Would they get more or less than one each? What if four students had three chocolate bars to share? Discourage students from doing the calculation in order to decide.

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Constructing Arrays

Have students construct arrays for a given number using grid paper, pegboards or blocks. Ask them to record each array using two multiplication sentences, such as 6 x 4 = 24, 4 x 6 = 24, and two division sentences, such as 24 ÷ 6 = 4 and 24 ÷ 4 = 6. Ask: What do each of the numbers show in the array? How is 6 x 4 = 24 the same as 4 x 6 = 24? How is it different? Can we say 24 ÷ 6 = 4 and 24 ÷ 4 = 6 are the same? Why not?

Unknown Number

Ask students to write open multiplication sentences to solve problems, and then say whether the unknown number is the number of groups or the amount in each group. For example: Tanya has five times as many marbles as Jill; if Tanya has 70, how many does Jill have? Or: Jill has five marbles; how many times as many marbles does Tanya have if she has 70? This could be written either as 5 x ☐ = 70 or as ☐ x 5 = 70. Ask students to read their number sentence to others and say how it represents the situation. Ask: What does the missing number represent in each case?

Fraction Problems

Have students use materials, such as paper or diagrams, to solve multiplication problems involving fractions. For example: If 32 students need half a piece of paper each, how many whole pieces of paper will be needed? Ask students to reflect on what they did and record this by writing a description of their thinking. Ask: Which sign can be used to show how you found your answer? How can you decide between multiplication and division? For example: Is it $\frac{1}{2}$ x 32 = 16 or 32 ÷ 2 = 16, or can it be both?

Multiplication and Division Stories

Have students review a collection of different multiplication and division stories (See Background Notes, page 25) and decide whether the unknown number represents the whole, the number of groups or the quantity in each group. Ask: If the whole is the unknown, which operation can be used to work it out? If the number of groups or the quantity in each group is unknown, which operation can be used? Why?

Halving Quantities

Students extend *More or Less*, on page 75, by using the calculator to work out the results of halving quantities. Ask: What keys can you use to find half of six? How can you multiply by a half using the calculator (multiply by 0.5)? How is 6 ÷ 2 the same as 0.5 x 6?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Small Units

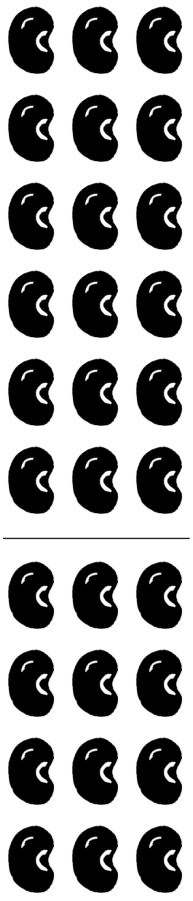
Extend *Jugs*, on page 76. Have students create groups to count large collections or measure quantities with small units. Present situations where students have to say how many dots there are in a large array, how wide the classroom is in centimetres, or how many eyedroppers of water there are in a jug. Ask students to decide how they could efficiently work out the amount without counting the single units by ones or repeatedly counting the number of units in each group. How would knowing the number in a row and the number of rows in the array help? How would knowing that a metre ruler is the same as 100 cm help? Encourage students to use multiplication to find out how many or how much.

Multiplication and Division

Encourage students to recognize the relationship between multiplication and division. Ask them to draw a 7 x 8 array, and then record multiplication and division number sentences about the array. Ensure that a range of examples are drawn out, such as 7 x 8, 8 x 7, 56 ÷ 8, 56 ÷ 7, $\frac{1}{7}$ x 56, $\frac{1}{8}$ x 56.

Arrays

Have students use materials, such as beans or cubes, to construct arrays then partition them in different ways and record a number sentence for each. For the 10 x 3 array shown left, they might write: *10 x 3 = (6 x 3) + (4 x 3) or 30 = (6 x 3) + (4 x 3)*. Ask: How can you check that your partner's number sentences match the array your partner has made without calculating the answer? Which numbers did you use—the number in each row or the number of rows?



10 x 3 = (6 x 3) + (4 x 3)

More Arrays

Extend *Arrays*, above, by posing problems such as: If you had four rows of three blocks and added some more rows to make 11 rows altogether, how many rows did you add? How many blocks did you add? If you had some rows of five and added four more rows of five to make 12 rows of five, how many rows did you start with? How many blocks did you start with?

Which Operation?

When solving the following problems, have students say what is missing—the whole quantity, the number of groups or the number in each group—to justify their choice of operation:

- The local hockey arena can hold 525 people and has 15 rows of seats. How many seats are there in each row?
- How many students can have 6 markers if there are 55 in the container?

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- If 28 students drink about $\frac{3}{4}$ of a litre of juice each, how much juice will have to be ordered for the party?

Ask: What operation is required when you know the number of groups and the number in each group, but not the whole? What operation(s) can be used when you know the whole amount and the number of groups, but not how many in each group? What operation(s) can be used when you know the whole amount and the number in each group but not how many groups? Why can some situations be represented with both division and multiplication?

Division Questions

Have students write division questions from multiplication problems. Begin with a multiplication situation such as: Shane's car travelled three times as far as Jenny's. Jenny's car travelled 2 m, so Shane's car travelled 6 m. Ask students to construct two division questions from this multiplication situation. What questions could you ask if you know the whole amount (Shane's distance), but don't know the number in each group (Jenny's distance)? What question could you ask if you know the whole amount, but don't know the number of groups (how many times more)?

Equivalent Operations

Encourage students to use equivalent number sentences to make calculating easier. For example: Isabella needed to work out 56 ÷ 8. She knows her multiplication tables well but does not know her divisions as well. How could Isabella use multiplication to help her solve the division sentence? Ask students to draw an array or a diagram to show which piece of information is missing (the whole amount, the number of groups or the number in each group) and explain why they can work it out either way.

Fractions

Use pizza-sharing activities where three pizzas are shared between four people to draw out the idea that if anything is shared between three people, each person gets one-third of it. Therefore, 2 ÷ 3 is the same as $\frac{1}{3}$ of 2. Have students demonstrate with materials (paper, fruit, modelling clay), diagrams and a calculator with fraction functions, that 4 ÷ 5 is the same as $\frac{1}{5}$ of 4 and $\frac{1}{4}$ of 3 is the same as 3 ÷ 4.

Factors

Set students a target number, such as 105, and ask them to use their calculator to test whether various numbers are factors. Many will use "guess and check", that is, to test whether 13 is a factor they will guess what you would multiply 13 by to get 105 and test it with their calculator. Challenge them to find an easier way and draw out that they can use the inverse relationship between division and multiplication. For example: 105 ÷ 13 = 8.07 so 13 is not a factor; 105 ÷ 15 = 7, so 15 is a factor.

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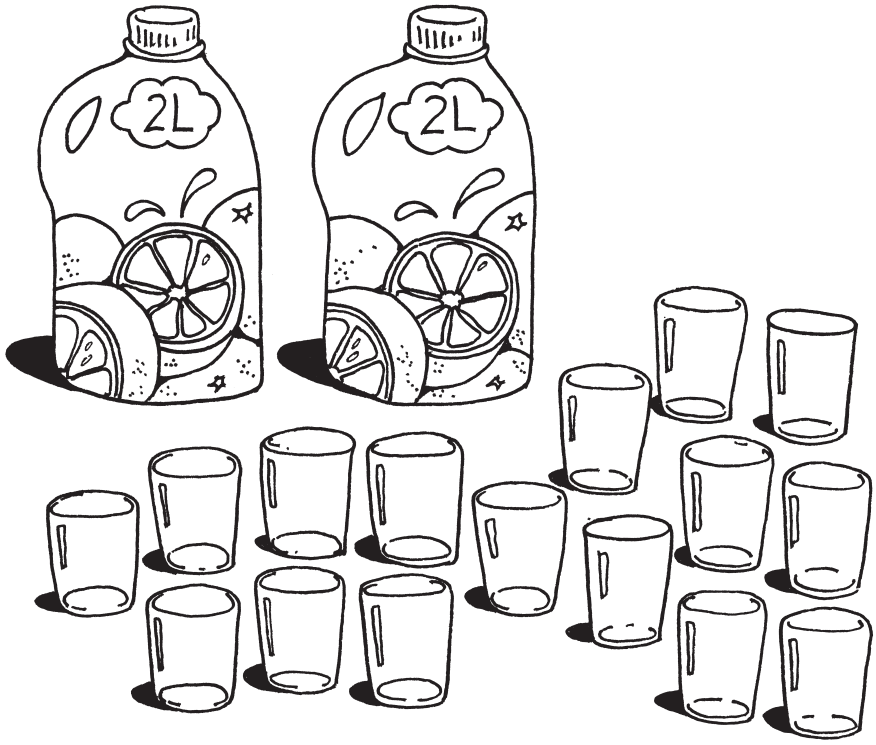
Grades 5-8: ★ ★ ★ Major Focus

Sharing Problems

Have students group to solve sharing problems. For example: Predict what each share will be if 420 paper clips are to be shared between 12 students. Ask: How many will I need to pull out for 12 students to get one paper clip each? What if I pull out another handful of 12, how many each? How many have I used up so far? How many groups of 12 will I need to pull out to share all the 420 paper clips? How many will each student have? Note that the number of groups gives the number in each share.

Rewriting Number Sentences

Pose problems with larger and more complex numbers so that students see a reason for rewriting a number sentence in a different but equivalent form. For example: Two 2-L (4000 mL in total) containers of juice were emptied into 16 large glasses. How much was in each glass? Students record the possible number sentences and say which they prefer to use and why: 16 x ☐ = 4000 or 4000 ÷ 16 = ☐.



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Did You Know?

Many students have a strong conviction that multiplication makes bigger and division makes smaller. This early and partial understanding is reasonable for multiplying or dividing by whole numbers, where multiplying does make bigger and dividing smaller. It can, however, develop into a misconception when students continue to believe and apply these rules in situations where they need to multiply or divide by fractional numbers less than one.

Target—A Diagnostic Game

How to Play:

- Player 1 enters any number onto the calculator.
- Player 2 has to multiply this by another number so that the answer will be as near to the target number, 100, as possible.
- Player 1 multiplies this new number, trying to get nearer to 100.
- The players take turns until one player hits the target by getting 100.------.

Player	Keys pressed	Display
1	64	64
2	x 1.5	96
1	x 1.2	115.2
2	x 0.9	103.68
1	x 0.9	93.312
2	x 1.08	100.7769
		WINNER!!

Many students playing this game for the first time will believe that Player 1 has lost as soon as he or she gets 115.2, since they cannot conceive of being able to "get smaller" through multiplying. The game itself helps students to overcome this misconception if you challenge them to find a way to continue.

Other students, when they have to act on a number such as 96, will think that, since they only need a "little bit" more, they need to multiply by a little bit. They may choose something like 0.4 and be very shocked to find the answer is 38.4. They expect the answer to always be bigger and in this case, be bigger by a little bit. They are probably thinking additively rather than multiplicatively.

Varying this game by dividing instead of multiplying to get as close as possible to 100 can also help overcome the related division misconceptions. This game is also an excellent tool for improving students' estimation skills.

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Key Understanding 6

The same operation can be said and written in different ways.

The concise way in which mathematics is written has many advantages. However, it makes it different from and, in some ways, more difficult to read and write than more narrative forms of text. Learning to read, write and say mathematics is an important part of learning mathematics. Students need to:

- relate special mathematical forms and symbols to everyday language
- recognize the various symbols used to represent the operations of addition, subtraction, multiplication and division
- know when and how the symbols are used. For example: $35 \div 7$, $\frac{35}{7}$ and $7\overline{)35}$ all mean the same thing, but $35 \div 7$ is not the same as $7 \div 35$

Students need to practise moving in both directions between the mathematically conventional symbolic forms and the various alternative everyday language forms, both oral and written, for example:

12 – 5	6 x 4
twelve take away five	four multiplied by six
twelve subtract five	six times four
the difference between 12 and 5	the product of six and four
5 from 12	six bunches of four
	six groups of four

Students who have reached the end of the Quantifying phase can use the range of alternative everyday expressions for addition and subtraction. Those at the Partitioning phase can do this for each of the four basic operations interchangeably.

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Sample Learning Activities

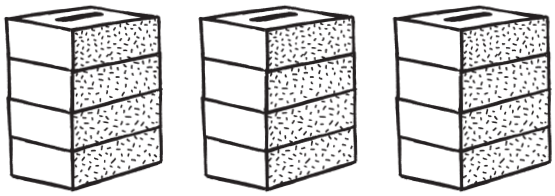
K-Grade 3: ★ ★ Important Focus

Packing Away

Focus students' attention on the words that describe the actions they use when packing away equipment and materials. For example: put with, took out, added, missing, two more, another, as well as, separate, subtract, plus, and, the same. Help students link their terms for joining, separating and equalizing to "add", "put together", "subtract", "take away", "make equal" and "make the same".

Equal Groups

Have students make a poster of things that occur in equal groups, such as rows, bunches, busloads, stacks, lines. Draw attention to different situations with the same numbers. For example, say: Three groups of four students make 12—that's three bunches of four; three stacks of four boxes make 12 boxes altogether—that's three groups of four as well.



Number Stories

Have students make up a story for a given number sentence, such as 6 x 4, 2 + 4, 6 ÷ 3, 6 – 5. Some students may need to be given a context. For example, after reading *Six Foolish Fishermen* (San Souci and Kennedy, 2000), students could write a story about the fishermen that is represented by one of those number sentences.

Two for Each

Pose problems that can be described as "two for each"; "two, two, two and two"; "that's four twos". For example: How can you show how many legs the four birds have? How can you show how many shoes in your group? How can you show how many eyes on the teddy bears? Have students draw or make models. Focus on "groups of".

Alternative Expressions

Have students use alternative expressions in questions. For example: You had $10 and you spent $6; $10 subtract $6; $10 take away $6; the difference between $10 and $6; $6 from $10.

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Math Language

Have students build a chart of language expressions for each of the operations: +, –, x and ÷. For example, for multiplication students might suggest *groups of, half of, times, multiply, groups of, twice as big*.

Math Word Wall

Operations			
Addition +	Subtraction –	Multiplication x	Division ÷
		groups of	
		times	
		half of	
		multiply	
		twice as big	

Read Aloud

Organize students to work in pairs. One reads aloud various number sentences from either a textbook or a card and the other writes the number sentence. Include examples such as 7 + ☐ = 27 and 12 = ☐ – 5. Compare the written version with the original to see if they are the same. If not, why? Create a list of the words used for the operation signs. Compare lists with other pairs to find more words.

Story Problems

Have students work in pairs. One reads aloud a number sentence from a textbook and together, they write a related story problem. Ask students to use different words for the operation by referring to the list made during the Read Aloud activity above.

Saying Symbols

Ask students to show problems, such as "How many 19s in 677?", in symbols, and read what they have written. Draw attention to how 677 ÷ 19 may be said as "677 divided by 19" but is also sometimes said "19 divided into 677" or "19s into 677". This is NOT written as 19 ÷ 677. (You can also link this to the other division symbol.)

In Other Words

When working on operations, model the use of different words. For example, use "add", "and", "plus", "with" when reading or saying number sentences involving the addition sign.

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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Which Goes with Which?

Provide students with cards that have symbolic expressions and phrases for division using the same pair of numbers. For example: 100 ÷ 5, 5 ÷ 100, $\frac{5}{100}$, $\frac{100}{5}$, $5\overline{)100}$, $100\overline{)5}$ five into a hundred, how many fives in 100, five divided by 100. Have students sort these into matching sets. Draw out that there are only two different divisions represented.

Matching Pairs

Ask students to write number sentences or symbols for situations such as: How much juice will each student in a group of eight get from a two-litre container? Which of these is correct: 2 ÷ 8, 8 ÷ 2, $2\overline{)8}$, $8\overline{)2}$, $\frac{2}{8}$, $\frac{8}{2}$? Draw out which ones match the problem and which do not, and the different order of the digits in the matching pairs.

Everyday Language

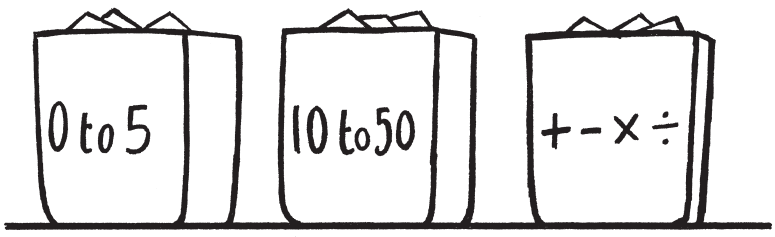
Have students read and use alternative everyday language to say what a number sentence means. For example: $3\overline{)18}$ and $\frac{18}{3}$ could each be expressed as "how many threes in 18" or "18 divided by three".

Chance Number Sentences

Play a game of chance to read number sentences. Organize students to work in groups of three. Have them make cards to go in three containers:

- one container has cards numbered from 0 to 5, including fractions and decimals
- one container has cards numbered from 10 to 50
- one container has cards showing the four symbols + , – , x and ÷

Place the three containers of cards in the centre of the group. Each student chooses a card from a different container and together they arrange the cards to make a number sentence. Each student says the number sentence in a different way, such as "3 times 12", "3 multiplied by 12."



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Key Understanding 7

Properties of operations and relationships between them can help us to decide whether number sentences are true.

By using the properties of operations and connections between operations, we can anticipate the general *effect* of operations on numbers without carrying out the particular calculations involved. This is important for two reasons. Firstly, it is an important aspect of number sense. For example, the fact that multiplying by a number less than one makes smaller means that you do not need to calculate to know that 32 x $\frac{1}{2}$ is less than 32, or that 0.3 x 0.2 must be smaller than 0.2 and so cannot possibly be 0.6. This helps us to decide whether results are reasonable and to pick up errors. Secondly, it is the basis of algebraic thinking and is therefore an important foundation for further mathematical progress. For example, using *properties of operations* and *relationships between them* means that we can construct and rearrange number sentences into simplified forms that help us solve equations and simplify computations.

Students should generate numbers, or pairs of numbers, that fulfill, some constraint, such as □ – ○ = 17 or □ x ○ = 72, without relying simply upon trial and error. For example: if 27 – 10 = 17, then 28 – 11 also equals 17 and so too does 29 –12, 30 – 13, 31 – 14 and so on. Students should consider whether they have all the possible numbers or pairs, how many there might be and how they could be sure that they have them all.

Students should also use their understanding of properties and relationships to:

- **complete** mathematical statements (without finding the "answer" to the calculations), for example:

  392 x 5 = □ x 392 (Put in the missing number.)

  14 ÷ 0.7 □ 14 (Put in < or = or >.)

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- **construct** mathematical statements, for example:

  5 x 26 = 5 x (20 + 6) = 5 x 20 + 5 x 6

  25 x 16 = 50 x 8 = 100 x 4 = 200 x 2 = 400

  I have to solve ☐ + 47 = 93. It's the same as 47 + ☐ = 93 and that's easier to work out!

- **check** the truth of mathematical statements, for example:

  273 x 5 = 1065 cannot be right because 273 x 5 > 260 x 5 = 130 x 10 = 1300.

  1024 x ☐ < 1000 can only be true if ☐ < 1.

  375 x 18 = 6745 cannot be right because the answer has to be even.

Many students interpret the = sign as "makes" or as a signal to "find the answer". Asked to complete ☐ + 7 = 12, they may place 5 in the box but nevertheless say that 12 is the answer. Some will think a sentence like ☐ + 5 = 12 + 3 is nonsense and others will place a 7 in the box. In drawing out this Key Understanding, it is important to emphasize that the = sign means "is equal to" and that it indicates that both sides of the equation represent the same number.

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Inverse Relationships

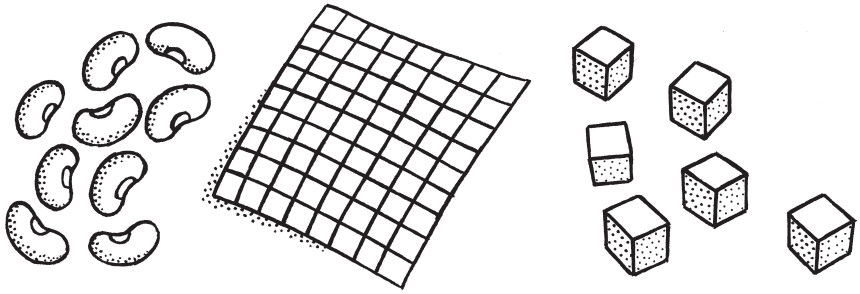
Read familiar stories to students, such as *Ten in the Bed* (Dale, 1999). Have students use the characters and events in the story to make up problems, such as ten in the bed and one fell out: 10 – 1 = 9. Then ask students to check the truth of the number statement by inverting the problem rather than working it out: adding one to nine to see if it is ten.

Addition

Revisit *Number Combinations*, on page 42, to show that numbers may be added in any order without changing the result. Ask students to represent and record all the combinations of a given number using everyday materials (beans, pop can pull-rings, straws, leaves), story contexts, and games. Display the addition number sentences and discuss, focusing on two at a time. For example, ask: Is 4 + 3 the same as 3 + 4? Use your beans to show how it is the same and how it is different. Ask: If Ben has three beans and Fran has four, will there be the same number of beans if they swapped, so that Ben has four and Fran has three? Why?

Missing Number

Have students use materials, such as beans, grid paper, cubes, to model open sentences, such as 14 + □ = 8 + ○, and say what the missing quantities have to be. Ask: What does the = sign mean?



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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Missing Number

Have students use inverses to find a missing number. For example: Use 327 – 118 = 209 to solve 327 – 209 = ☐ ; 118 + 209 = ☐ ; 209 + ☐ = 327; 118 = ☐ – 209 and so on. Ask: How do you know what the numbers must be? (Choose large numbers that discourage students from calculating the answers.)

Inequality Statements

Ask students to write inequality statements, such as 4 + 16 < 5 + 16, and show each side using materials. Ask them to decide whether statements involving large numbers, such as 197 + 385 < 200 + 400, are true without using materials or calculating.

Equals

Have students read the = sign in open number sentences as "is equal to". For example, 18 = ☐ + 13 is read as, *18 is equal to something add 13*; ☐ + 6 = 14 + 2 is read as, *something add six is equal to 14 add two*. Ask: What does the = sign mean? What does the missing number have to be?

Extension Activity

Extend the previous activities, and ask students to create complex open sentences for others to solve, such as 14 + ☐ = ◯ – 25. Ask: How can you make both sides of the = sign equal the same amount?

Choose Your Numbers

Write numbers on the board, for example:

12	20	21	5	16
4	8	13	26	18

Ask students to choose any three of these numbers to write a true addition or subtraction sentence. Have them share their number sentence. Ask: Does it work? Could you use your numbers again to write other addition or subtraction number sentences? Repeat this activity using multiplication and division.

Today's Number Is...

Have students write different number sentences to make the same total. Encourage them to use what they know about the operations instead of calculating each example separately. For example: If we know 117 – 100 = 17 and 118 – 101 = 17, what would the next one be?

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Grades 3-5: ★ ★ Important Focus

Bigger or Smaller

Ask students to complete number sentences like the following by writing "bigger than" or "smaller than" without carrying out the calculation.

19 + 32 is ..... 20 + 32
19 + 32 is ..... 19 + 30
47 + 49 is ..... 50 + 50
47 + 49 is ..... 45 + 45
156 – 19 is ..... 156 – 20
156 – 19 is ..... 160 – 19

Ask them to justify their reasoning to a partner then check the calculations and revise their thinking if needed.

Easy Check

Extend *Bigger or Smaller*, above, by asking students how they could use sentences like those to estimate answers. For example: 47 + 49 is less than 50 + 50 and more than 45 + 45, so it is between 90 and 100. Ask: How could this help you check your work? What would you say to a friend who did this:

$$\begin{array}{r} 47 \\ +\ 49 \\ \hline 86 \end{array} \quad \text{or this} \quad \begin{array}{r} 47 \\ +\ 49 \\ \hline 816 \end{array}$$

Bigger, Smaller or Equal 1

Extend activities such as *Bigger or Smaller*, above, to include "equal to" examples formed by rearranging parts. For example:

47 + 49 + 3 is ..... 47 + 3 + 49
56 + 37 is ..... 56 + 4 + 33

Bigger, Smaller or Equal 2

Extend *Bigger, Smaller or Equal 1*, above, to include "equal to" examples formed by compensating. For example:

47 + 49 is ..... 46 + 50
56 – 19 is ..... 57 – 20
72 + 73 + 74 is .....73 + 73 + 73

Have students generate sets of equivalences for 47 + 49. Ask: Which changes to the number sentence help calculation? Help students describe the difference between the addition and subtraction lists. Ask: How do you change the numbers for addition? Is this the same or different for subtraction? Think of a simple rule for each.

Multiplication and Division

Repeat the *Bigger, Smaller or Equal* activities above for multiplication and division.

47 x 19 is ..... 47 x 10
47 x 19 is ..... 47 x 20
400 ÷ 19 is ..... 400 ÷ 10
400 ÷ 19 is ..... 400 ÷ 20

Checking Work

Give students a set of calculations completed by an imaginary student and ask them, without doing the full calculations, to find those which cannot be right. Include errors commonly made by students.

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

What Do You Know?

Have students use relationships between operations to answer: If we know that, what else do we know? Ask them to work out: 41 + 41 + 41 = 123. Challenge them to find other number sentences that say the same thing in another way, such as 123 ÷ 3 = 41, 123 ÷ 41 = 3, 3 x 41 = 123, 41 x 3 = 123, 123 – 41 – 41 – 41 = 0.

Broken Keys

Invite students to use connections between the operations to deal with a broken calculator key. For example: The division key on your calculator has broken. How can you find the answer to 210 ÷ 7?

Calculating Strategies

Ask students to check the truth of calculating strategies. For example: Sue said to calculate 57 + 99, she says 56 + 100 and, to calculate 58 x 99, she says 57 x 100. Is she correct? Why or why not?

Bigger or Smaller

Extend *Bigger or Smaller*, on page 90, to include decimals and fractions.
For example:

4.73 + 3.56 ..... 4 + 3
4.73 + 3.56 ..... 4.5 + 3.5
4.73 + 3.56 ..... 5 + 4

$\frac{2}{3} + \frac{3}{4}$ ..... $\frac{1}{2} + \frac{1}{2}$
$\frac{2}{3} + \frac{3}{4}$ ..... 1 + 1
$\frac{2}{3} + \frac{3}{4}$ ..... $\frac{3}{4} + \frac{3}{4}$

Bigger, Smaller or Equal

Extend activities such as *Bigger or Smaller*, on page 90, to include "equal to" examples formed by rearranging factors.
For example:

25 x 19 x 4 ..... 25 x 4 x 19
25 x 36 ..... 25 x 4 x 9

Multiplication and Division

Extend *Multiplication and Division*, on page 90, to include decimal and fraction multipliers and divisors.
For example:

236 x 1.3 236
236 ÷ 1.3 236
236 x 0.3 236

236 ÷ 0.3 236
236 x $\frac{3}{4}$ 236
236 ÷ $\frac{3}{4}$ 236

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Grades 5-8: ★ ★ ★ Major Focus

Equivalent Statements

Ask students to construct equivalent statements by multiplying or dividing both sides of a number sentence. For example: Continue: 64 x 124 = 32 x 248 = 16 x 496 = ☐. Ask: What is happening to the multiplication sentences? Write a rule about what you have discovered. How does knowing this rule help you work out 5 x 384? Why does it work?

Number Line

Have students use a number line to predict the results of operating on decimals, including those less than one. For example: Draw a large number line from zero to five. Randomly place points labelled A, B, C and D between zero and two. Ask: What numbers could the letters represent? Have students work out approximate solutions to "letter sentences", such as A + B, D ÷ A, C x D and so on. Ask: What helped you to do this? When did you know that an answer was going to be larger or smaller?



Target Multiplication

Use calculator games to show that multiplication does not always make numbers bigger. For example, play the *Target* game described in Did You Know? on page 81. Have students play the game and after several rounds, ask: Does multiplying always make numbers bigger? What do you have to multiply by to make your number smaller? What do you multiply by to make your number bigger? Write the rule so you will remember next time you play the game.

Target Division

Use calculator games to show that division does not always make numbers smaller. Play the *Target* game described in Did You Know? on page 81, but use division and make the target number 30. Vary it by changing the target numbers. Ask: How does the size of the numbers affect the size of the answer? Write the rule so you will remember next time you play the game.

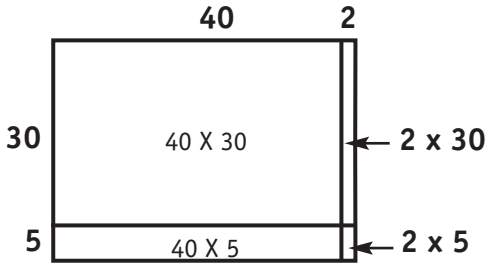
The Same As

Have students use number strips to interpret the = sign as "is equivalent to" or "is the same as". Begin by writing a number sentence, such as 41 + 41 + 41 = on the left side of a long strip of paper. Ask students to write as many equivalent number sentences, such as (3 x 40) + (3 x 1) = ☐, as they can before writing the answer. Ask students to share their strips by reading their equivalent sentences to others.

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Equivalent Statements

Have students use an array representation of a multiplication sentence to construct equivalent statements that will make calculating easier. For example:



Students divide the grid into sections and label the multiplication needed for each section. Compare number sentences and representations for 42 x 35. Is it easier to calculate a 42 x 30 and a 42 x 5 section or 40 x 30, 40 x 5, 2 x 30, and a 2 x 5 section. Ask: Are there other ways to section your array to make calculating the product easier? (See Background Notes, pages 25 to 28.)

True or False

Ask students to decide why number sentences cannot be true, without doing the calculations. For example: Why is 0.3 x 0.2 not 0.6? (It has to be smaller than either.) Why is 1.2 x 100 not 1.200? (It has to be bigger than 1 x 100.) Why is 3.0 ÷ 10 not three? (It has to be smaller than three.) Why is 3.05 ÷ 10 not 3.5? Why is 1.23 + 3.4 not .157? Ask: What might a student have been thinking to give those answers?

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Key Understanding 8

Thinking of a problem as a number sentence often helps us solve it. Sometimes we need to rewrite the number sentence in a different but equivalent way.

To solve everyday problems involving numbers, we have to represent the problem as a number sentence. For example, when we buy two things that cost $35 and $17, we have to think of it as "35 add 17 is what?" Students need a lot of experience in representing problems in ways that enable them to deal with the problems mathematically. However, for common situations, the process will eventually become almost subconscious.

In some cases, the number sentence we use is a standard one that shows what operation we have to do, such as 16 + 7.3 =; 22 x 4 =; 105 ÷ 5 =; 3.9 – 2.4 =. That is, we choose the operation and the solution can be found right away by a calculator (machine or human). The reason we want students to understand the operations is so that they can choose the right one to apply in a range of situations.

In other cases, the number sentence we choose has an unknown quantity embedded within it. For example:

The 2.5-kg package of hamburger meat cost $12. How much per kilogram is that?

Considering this problem, a student might think: *What did they multiply 2.5 by to get to 12?* or *2.5 times what gives 12?*, which can be written as 2.5 x ☐ = 12. Here the "missing multiplier" cannot be found by entering the number sentence into a calculator or doing a single mental or written computation. Nevertheless, the number sentence represents the problem and can be solved (See Background Notes, page 25). We could solve it by trial and error or rearrange the sentence. For example, we could use the relationship between multiplication and division to rewrite the sentence: if 2.5 x ☐ = 12 then ☐ = 12 ÷ 2.5. This process is the basis of generalized (or algebraic) thinking and its development should be a high priority (See Key Understanding 7, page 86).

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Of course, another student might think of the problem immediately as a division, seeing that the price per kilogram is the total cost divided by the number of kilograms (12 ÷ 2.5). It is important that students see these two ways of thinking about the problem as equivalent so they can choose the best approach (See Background Notes, pages 26 to 27). This flexibility depends on students believing and understanding the following:

- Two number sentences are equivalent (that is, mathematically the same) when they represent the same situation.
- It is not necessary to go back to the original situation to know that two number sentences are equivalent; instead we can use properties of operations and relationships between them.

Students need to practise representing problems in reduced symbolic forms. Sometimes, it is complex. For example, the statement "Dave is six years older than Kim" does not have any obvious linguistic cues to indicate what operations are involved. Students need to be able to say to themselves: *If you take Kim's age and add six, you have Dave's age, so Kim's age* + 6 = *Dave's age*. Many students will follow the sentence sequence and incorrectly write: *Dave's age* + 6 = *Kim's age.* They need to learn to make sense of the whole expression and be discouraged from using cue words or translating word for word. (See Background Notes, pages 26 to 29.)

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Money

Re-price items in a store flyer into whole dollars. Nominate an amount of money for students to spend and have them choose items from the flyer. Ask them to use number sentences to show how much they have, what they have bought and what they have left to spend.

Identifying Operations

Have students listen to word problems and identify the operation in the problem. For example: Fran had seven stickers. Tom gave her some more. Now she has 19. How many did Tom give Fran? Ask students to hold up an "add" or "take away" sign to show what kind of problem it is. Ask: What was the clue that made you think it was an add (take away)? Could we write an addition number sentence for it?

Inverse Relationships

Organize students into pairs so they can practise writing open number sentences using inverse relationships. One student secretly enters a number into a calculator and adds an agreed upon number, such as five. The other student writes the number sentence used, such as ☐ + 5 = 9, and rewrites it in a way that can be solved on the calculator, such as 9 – 5 = ☐.

Did You Know?

Students require real experience in deciding what operation is needed. Practise on "word problems" might only appear on pages labelled, say, "Addition". Or, there may be other obvious cues so that students do not need to think much about the situation or really choose the operation. In daily life, there may be extraneous information and we have to choose the relevant numbers as well as the operation. Ask students to do the following activity (See Appendix: Line Master 5).

Put a ✓ beside the problems for which 379 – 280 would give the answer. Give reasons for your choices.

- ❑ 379 students had to have their school photo taken. 280 students had been photographed by lunchtime. 80 of the students were in grade 6 and 200 were in kindergarten. How many students still have to be photographed?
- ❑ There was 379 kg of dog food in the factory's freezer. 280 kg of it was sold to local shops. How much dog food was left over?
- ❑ 379 posts are needed for one fence and 280 posts are needed for another. How many posts are needed altogether?
- ❑ Maxine finished the car rally in 379 minutes. Her older sister, Jane, finished the rally in 280 minutes. How much longer did Maxine take?

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Signs

Ask students to hold up a sign that shows the appropriate operation as word problems are read out. When some students show the x sign while others show the + sign, for example, discuss the differences that arise. Encourage students to justify their choice. Ask: Could they both be right?

Classroom Situations

Have students write number sentences for situations that arise in the classroom. For example: Everyone needs half a sheet of paper. How many pieces of paper do we need? Ask: How many different ways can this be written as a number sentence?

Best Buy

Have students use a calculator to determine which of several sizes of an item would be the best buy. For example: Is it cheaper to buy two small bottles of shampoo or one large one? Ask students to justify the operations they choose.

Choose the Operation

Ask students to match word problems with given operations. For example: 105 + 7, 105 x 7, 105 – 7, 105 ÷ 7:

- Seven airplanes carried 105 passengers each. How many passengers altogether?
- Out of the 105 passengers, seven had never flown before. How many had flown before?
- 105 cows were put into small pens of seven. How many pens were needed for all of the cows?
- A farmer bought seven cows to put with his herd of 105. How many cows does he have now?

Discuss with students the information in each problem that signalled the correct operation.

Numbers and Signs

Have students use numbers and signs to solve problems and compare their strategies with others. For example: If one penguin ate 18 fish and another ate 22, how many more fish did the second penguin eat? Discuss the different number sentences created and how each is related to the original situation. For example, ask: How is 18 + ☐ = 22 the same as 22 – 18 = ☐? Which sentence would you use on the calculator? Why?

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Grades 3-5: ★ ★ Important Focus

Fractions and Operations

Extend *Choose the Operation*, on page 97, to include fractions. For example, sharing out three pies among four students. (See *Sharing Diagrams*, page 65.) Ask: How is 3 ÷ 4 the same as ☐ x 4 = 3? Which sentence would you use on a calculator? Why?

Classifying Problems

Have students look at problems that can be solved using either addition or subtraction and the two number sentences generated for each. Ask: How are the sentences the same? Which operation is most helpful if you were solving these problems with a calculator? Which would you use if solving it in your head? Why? (See *Classifying Problems*, pages 37 and 46.)

Related Numbers

Give students three related numbers and ask them to write as many number sentences as possible. For example, for the number three, six and 18 students might write 3 x 6 = 18, 18 = 6 x 3, 3 = 18 ÷ 6 and so on. Ask: If the 18 is unknown, which of these can be solved using a calculator? Which cannot? Why? What if the three is unknown, can you solve the same ones with the calculator?

Clarifying Problems

Have students rephrase a problem to clarify it. For example: Casey earns $4.50 per hour for working between 9 a.m. and 5 p.m., and $5.50 per hour when working after 5 p.m. How much did she earn when she worked from 2 p.m. to 10 p.m? This can be rephrased as: Casey worked three hours at $4.50 per hour and five hours at $5.50; how much did she earn? Ask students how they arrived at a number sentence from the rephrased problem and which sentences are easier solved with a calculator. Ask students to compare their different sentences and say how each is the same.

Story Problems

Invite students to choose a number sentence that could be used to solve a story problem. For example: Jenny finished the car rally in 369 minutes, while her brother Phillip finished it in 283 minutes. How much longer did Jenny take?

369 + 283 = ☐ ☐ = 369 – 283 283 + ☐ = 369
369 – 283 = ☐ 369 + ☐ = 283

Ask: Why can the number sentence(s) be used to solve the problem?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

Equivalent Sentences

Have students use equivalent number sentences to make calculating easier. For example: Christine needed to work out 56 ÷ 8. She knows her multiplication tables well but is not fast at division. How could Christine use multiplication to help her solve the division sentence?

Problem Solving

Ask students to use different but equivalent number sentences to solve a problem. For example: Two students were trying to solve Miles's problem. He was saving for a CD player costing $169. He had $97 and he wanted to know how much more he needed. Guy worked on this calculation: 97 + ☐ = 169, while Miles had this: 169 – 97 = ☐. Ask students to explain to their partners why the calculations were chosen and whether they can both be correct.

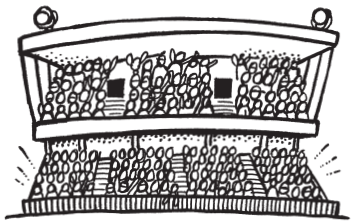
Larger Numbers

Pose problems with larger and more complex numbers so that students see a reason for rewriting a number sentence in a different form. For example: A 2-L container of juice was emptied into six large glasses. How much was in each glass? Ask students to record the possible number sentences and say which they prefer to use if an approximate answer is required: 6 x ☐ = 2000 or 2000 ÷ 6 = ☐.

Rewriting Problems

Have students rewrite a problem in their own words. Then ask them to progressively find shortcuts to write it: first with words, then with words and symbols, and finally, with symbols alone. For example, for the problem, "The grandstand at the show can hold 525 people and has 15 rows of seats. How many people can sit in each row?", a student might write:

- *The grandstand holds 525 people in 15 rows. How many in each row?*
- *That's 15 rows of how many people are in 525.*
- *15 groups of _____ = 525*
- *15 x ☐ = 525*
- *525 ÷ 15 = ☐*



How Much? How Many?

Repeat *Rewriting Problems*, above, with more complex problems, such as:

- Jesse paid for his computer game on a monthly payment program, with a $12 deposit and five $16 payments. How much was the game?
- The bus for the excursion will hold 77 adults. If three students can take the place of two adults, how many students will the bus hold?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

Sorting Sentences

Extend the *Rewriting Problems* and *How Much? How Many?* activities on page 99 to a wider range of problems. Present a problem: Guy's mom told him to cook a roast. She said it needed to cook for 20 minutes, and then 30 minutes for each kilogram. The roast weighed 4.5 kg. How long should Guy cook it for? Have students write a number sentence for the problem on a card. Collect, sort and display the variety of number sentences. Ask students to explain how they paraphrased the problem in their head to come to their number sentence. Ask: Which number sentences will solve the problem?

Matching

Have students decide which number sentences match a word problem. For example: Amanda works at the supermarket for an hour a day for five days of the week and four hours on Saturdays. The pay is $5.50 per hour during the week and $7 per hour on Saturdays. How much does she earn?

(7 + 5.50) x (4 + 5) (4 x 5) + (5.50 x 7)
(5 x 5.50) + (4 x 7) (7 x 5) + (5.5 x 4)

Checking Solutions

Invite students to use a different but equivalent number sentence on a completed problem as a good way to check solutions. For example:

- The lawn mower cost $467. Jemma only had $275. How much more did she need? 275 + ☐ = 467. Answer 192.
- I have a photo that is 75 cm high, but I want to put it into a photo frame that is only 15 cm high. By how much will I have to reduce my photo? 75 x ☐ = 15. Answer $\frac{1}{5}$.

Ask: What number sentences could you key into your calculator to check that your solution to each problem is correct?

Everyday Problems

Have students ask mathematical questions from everyday situations and write number sentences to fit the questions. For example: The grade 6 class had a 25-L container of ice tea to sell to raise money for their school camp. They wondered if they would make more money by using 300-mL cups and charging 50 cents per cup, or by using 200-mL cups and charging 20 cents per cup. The cups were donated. Ask students to list the questions that need to be answered, along with the number sentences required to answer each question.

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Choose the Operation

Have students match word problems that use the same pair of numbers to different operations. For example: Match 23.5 + 13, 23.5 – 13, 23.5 x 13 or 23.5 ÷ 13 to a set of six or seven problems that use 23.5 and 13 (See Did You Know?, page 96).

Exploring Word Problems

Ask students to decide which word problems match a given number sentence and explain why. For example, 12.6 ÷ 3:

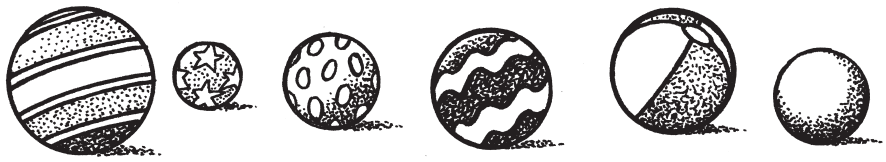
- 12.6 L of juice is shared between three groups. How much for one group?
- If you have 12.6 L of juice, how many groups could have 3 L each?
- A marble rolls 12.6 m in three seconds. What is its average speed in metres per second?
- If the marble travels at 12.6 m per second, how long will it take to travel 3 m?
- Eva put 12.6 mL of red food colouring into a test tube. This was three times as much as the blue. How much blue did she put in?
- A picture has been enlarged three times its original size. If it was originally 12.6 cm high, how long is it now?

How Old?

Have students write shortened sentences that combine numbers and words to represent simple relationships between two variables. For example: Maria is two years older than Sam; Andrew is two years younger than Heather; Cameron is twice as old as James.

Unknown Quantity

Ask students to write number sentences to match more complex "unknown quantity" problems. For example: Two teams of students had 24 balls to share between them. The teacher said that the bigger group was to get six more balls than the other group. How many balls should each group get?



Solving Problems

After solving problems like those in the *Unknown Quantity* activity above, challenge students: You have been asked to write instructions to another class on how to solve problems like this. Their problem will have different numbers, so you have to explain how to solve the problem without using any numbers.

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Key Understanding 9

We make assumptions when using operations. We should check that the assumptions make sense for the problem.

Students should think about the assumptions they need to make in order to use an operation. It should be treated incidentally, and questions like those in the Sample Learning Activities asked regularly as students go about their everyday mathematical work. That is, on the whole, special activities will not be needed. Students should be asked questions, such as: Does multiplying make sense here? What would we need to assume? Will two people really cost twice as much as one? If it is not likely to be exactly right, will it be close enough?

To decide whether using an operation makes sense, students need to understand the operation. For example, they may know that 18 students in the class like pizza and 14 like hot dogs. But this does not mean that 32 students will be happy so long as both pizza and hot dogs are provided—as few as 18 might be! Students need to understand that when we add a number in one collection to a number in another collection, the answer will be the number in the combined collection *only* if the two original collections had no overlap. Another example of having to be careful in adding relates to the different uses of the fraction symbol. When we write $\frac{7}{10}$ on a test paper we say "seven out of ten" and really think of it as two numbers. If we get $\frac{7}{10}$ for Section A and $\frac{8}{10}$ for Section B, we cannot add the numbers $\frac{7}{10}$ and $\frac{8}{10}$ to get the total test result because the two ratios do not show fractions of the same whole. Rather, we add the number right in each section (7 + 8), and the possible number of marks in each section (10 + 10) and record the two results as $\frac{15}{20}$. Students need to understand that adding fractions assumes that the two fractions are each a single number that show parts of the same whole.

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Students also need to think about what is sensible or realistic in real life. How well they are able to judge this will depend on how familiar they are with the context in which a problem is set. For example, in deciding whether to multiply, students may need to decide whether it makes sense to assume that it would take ten times as long to run 1000 m as it would to run 100 m. They may think about their own running, or they may do some research and investigate the pace set by sprinters, middle-distance runners and marathon runners.

Thinking about and checking assumptions are important components of learning how to select and use appropriate strategies and operations to solve problems. Students do not understand the meaning of an operation unless they can tell when they can and cannot use it.

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Rainy Day

Pose an everyday problem, such as: There are 12 raincoats and four umbrellas outside. How many people came to school ready for the rain today? Ask students if adding the raincoats and umbrellas will tell them how many students were ready for the rain. Ask: What could we be sure about?

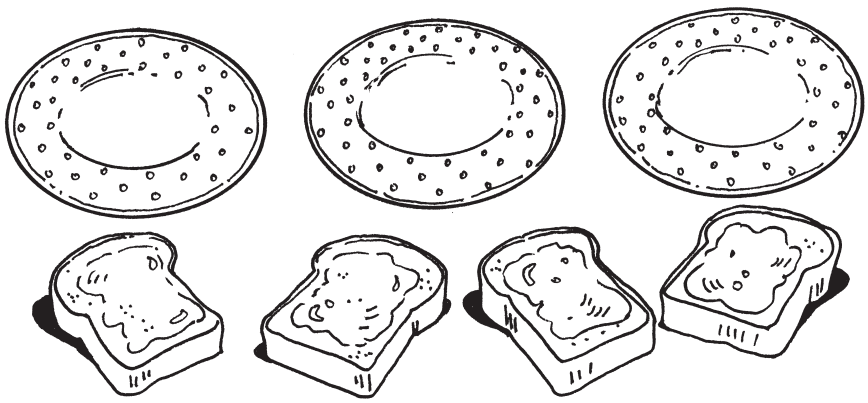
Graphs

When students are graphing, ask questions that focus on whether categories overlap. For example, when graphing pet ownership, ask: How many students have cats or dogs? Some students are likely to add the groups of pets together to find the total. Display the results and ask students to stand if they have cats or dogs at home. Ask: How can that be? Your answer was 18 but only 15 students are standing. Is something wrong?

Remainders

When sharing snacks, have students decide how to deal with remainders. For example:

- When sharing out raisins between five students, would it make sense to cut up the two raisins left over? What should you do with them?
- What could you do if you had four pieces of toast to share between three children? What could you do with the remaining piece of toast?



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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Remainders

Ask students to investigate ways of dealing with remainders. For example:

- Share 48 cans of juice between 23 students.
- 136 students are going on a field trip. How many buses do we need if each one carries 50 students?
- You need 9 m of fabric to make 6 flags. How much fabric for each flag?

Have students discuss what to do with the remainder in each case.

Venn Diagram

Have students produce a Venn diagram to show the sports they play after school. Ask: Can we find the total number of students playing sports after school by adding all of the numbers? How can we find what we are looking for when the information provided overlaps?

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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Correct Operations

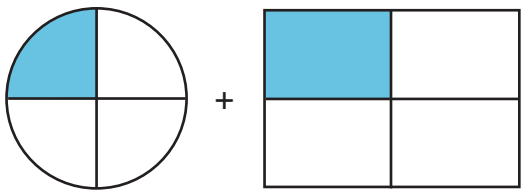
Ask students to identify "hidden" assumptions in using an operation and decide whether they are sensible. For example, Julia used multiplication for these problems:

- A man can run 1 km in three minutes. How long will it take him to run 6 km?
- One 125-g jar of coffee costs $3.45. How much will a 1-kg container of the same coffee cost?

Ask: What did Julia assume? Are the assumptions reasonable? Can you give a more realistic answer?

Sense or Nonsense?

Have students decide if it is sensible to carry out operations in situations such as the following. Ask: In this illustration, would it make sense to add the shaded parts? Why? Why not?



(NB: It is not sensible!)

If you got half of last week's spelling words correct, and a quarter of this week's spelling words correct, would it make sense to add $\frac{1}{2}$ and $\frac{1}{4}$ and say you got $\frac{3}{4}$ of your spelling words correct altogether? Why? Why not? How could you find out what fraction of words you got correct altogether? (NB: You would need to find the total number of words you got correct in the two tests and write that number over the total number of words tested. This ratio tells you the fraction you got correct altogether.)

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Chapter 3

Computations

Choose and use a repertoire of mental, paper and calculator computational strategies for each operation, meeting needed degrees of accuracy and judging the reasonableness of results.

Overall Description

Students are justifiably confident of their capacity to deal, correctly and efficiently, with everyday counting and computational situations. They can count a collection one-to-one, recognize skip counting in twos or threes as more efficient, and combine collections using strategies, such as counting on.

They know the addition facts to 10 + 10 and multiplication facts to 10 x 10. They extend these with a flexible repertoire of mental strategies for each of the four operations on whole numbers, money and simple fractions. They use written approaches as a backup for calculations they cannot store completely "in the head". These may include diagrams, jottings, standard routines, and supporting technology for students with disabilities.

They understand that calculators or computers are the sensible choice for repetitive, complex or lengthy calculations. They use them efficiently, correctly interpreting calculator displays.

Students judge the appropriate level of accuracy, are accurate when necessary, and otherwise estimate and approximate.

As a matter of course, they check that the results of their computations make sense both in terms of the numbers and operations involved and the context in which the calculations arose.

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BACKGROUND NOTES

Learning Basic Facts

Students should not be expected to try to memorize facts they do not understand. Equally, however, understanding where the basic facts come from and having worked them out for themselves is NOT enough to enable students to remember them. Students usually do need some drill with number facts if they are to be able to readily recall them. What is needed is a rational rather than a rote approach to learning the basic facts.

Addition Facts

Students might first discover and record the addition combinations to ten, convincing themselves that particular number facts "always work". They can often do some of these by counting in their mind's eye (three and two more) and/or quickly checking using their fingers, as well as by using materials. They should develop organized lists showing the numbers that fit together (part-part-whole) to make five or eight or ten. Once there is a meaningful basis for these "facts", students need focused practice on remembering them—small amounts at frequent intervals.

Meanwhile, many of the combinations to 20 should be being established using materials and diagrams. Students should be using mental arithmetic to extend the facts they already know and remember. For example: *I don't know 7 + 5 but it is*  *, which is two more than 5 + 5, so it must be 12; 8 + 6 is like two sevens so it must be 14.*

Addition Facts

+	1	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10	11
2	3	4	5	6	7	8	9	10	11	12
3	4	5	6	7	8	9	10	11	12	13
4	5	6	7	8	9	10	11	12	13	14
5	6	7	8	9	10	11	12	13	14	15
6	7	8	9	10	11	12	13	14	15	16
7	8	9	10	11	12	13	14	15	16	17
8	9	10	11	12	13	14	15	16	17	18
9	10	11	12	13	14	15	16	17	18	19
10	11	12	13	14	15	16	17	18	19	20

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Once the combinations to 20 have been discovered and recorded, students should be introduced to the use of a two-way table to record the facts (See Appendix: Line Master 18).

Students will need to learn how to read the table and should investigate patterns in it. They should note the sums on either side of the diagonal are in a sense "the same" and that the number of facts to be remembered is almost halved when we use the commutative property of addition (55 instead of 100). The "double numbers" on the diagonal (2, 4, 6, 8, 10, ...) are helpful in a lot of contexts and students' attention should be drawn particularly to this sequence of numbers.

Multiplication Facts

The usual approach to learning multiplication facts is to learn to chant through the multiplication facts in order. Students learn the two times "table", the three times "table", then the four times "table", then the fives, and so on. While students do need to memorize the basic multiplication facts, learning them by chanting tables is not a particularly helpful approach, for the following reasons.

- Firstly, many students who have learned their "tables" in this way have difficulty remembering the facts without chanting through the table. Hence, it hinders the development of instant recall rather than helping it.
- Secondly, setting out the multiplication facts in columns and learning each set of tables separately masks the commutative property. Therefore, many students who *do* know, say, six fives, do not relate it to five sixes and have to remember almost twice as many facts as they need to.
- Thirdly, other patterns, such as that six times is double three times, are masked, which also increases the memory load for students.

It is likely to be much easier for students to remember basic facts if they practise them in clusters that help work them out. A possible sequence could be as follows:

Build up the facts to 5 x 5

Start with the twos (doubles), fours (double doubles), fives (because of the easy patterns and the links to our fingers) and then the threes. Put these in a two-way table. Commutativity reduces the 25 facts to be remembered to only 15, and if we remove the ones, there are only

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ten to remember. Use the 5 x 5 table to show students that by learning just ten facts they "get" 25! Consolidate these facts and build up speed of recall with frequent short periods of practice.

X	1	2	3	4	5
1	1	2	3	4	5
2		4	6	8	10
3			9	12	15
4				16	20
5					25

Note that students may, at the same time, be able to skip count in twos, threes, fours and fives well beyond these facts. However, this requires them to work their way through the skips. This is not the same as being able to immediately recall 4 x 3, which is what focused drill should help them to do.

The ones and twos

At this stage, students could focus on the notion of "doubling" and build their capacity to readily find "double a number". (This should be linked to the diagonal of the addition table and to the notion of even and odd numbers in order to help students make connections between related mathematical ideas.)

The tens

Focus on groups of ten and counting in tens. Help students become convinced about why three tens are the same as ten threes and practise these together. There will be ten facts to remember, but the pattern makes them easy and students will readily recall them.

The squares

Many teachers also find that explorations of patterns of squares help students learn the squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. It seems students "like" the square numbers and learn them fairly readily. Thus, facts such as 7 x 7 may be learned earlier than other related facts. Students should develop instant recall of these facts.

Students have now developed the facts from 1 x 1 to 5 x 5 and the facts involving one, two and ten and the squares. Help them make a two-way table (See Appendix: Line Master 19) in which they record the multiplication facts they now know.

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Multiplication Facts

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15					30
4	4	8	12	16	20					40
5	5	10	15	20	25					50
6	6	12				36				60
7	7	14					49			70
8	8	16						64		80
9	9	18							81	90
10	10	20	30	40	50	60	70	80	90	100

There are various orders in which the remaining "facts" can be developed. Some teachers find it most helpful to move out in an ever-increasing square, so that the next cluster of facts is those to fill in the square to 6 x 6. This requires the addition of three new facts (6 x 3, 6 x 4, 6 x 5) and their "partner" facts (3 x 6, 4 x 6, 5 x 6). Then move out to 7 x 7 and so on.

An alternative sequence could be as follows.

The five facts

Build up the five facts to 5 x 10, noting the relationship to the ten facts (five eights is half of ten eights; half of eight tens is four tens or 40; so 5 x 8 is 40) and the pattern in the units digits. Add to the table and memorize.

The four facts and eight facts to 8 x 8

First build on the "doubles" or twos (2 x 1 to 2 x 10) to get the "double doubles" or fours (to 4 x 10). Add these to the table. Double the four facts to produce the eights up to 8 x 5 (if you know 4 x 3, you can double to get 8 x 3). Add these to the table. Use commutativity to work out the additional facts (if you know 8 x 3 you also know 3 x 8). When first practising these facts, give students plenty of time to work them out mentally using the doubling strategy or some other method they prefer. Gradually build up speed to get "instant recall". Use various doubling and other patterns to build up the extra eight facts (8 x 6 to 8 x 8). Practise to memorize.

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The three facts and six facts

Build up the three additional three facts (3 x 6, 3 x 7, 3 x 9) and add to the table. Double these to get the sixes or use other known facts (six fours is five fours and four more). Five new facts to learn and their partners give you ten more.

The nine facts

Build on the threes and the sixes using number partitions to add the additional two nine facts and their partners. Although most are known, revisit the nines to link them to 10 – 1, so that students see that 9 x 7 is ten sevens take away seven. Initially, allow students time to do the calculation using mental arithmetic strategies and gradually help them memorize for speedy recall.

The seven facts

The seven facts are all known!

This is not the only possible sequence. The important thing is to assist students to use rational thought processes rather than rote memory to learn the facts.

Reducing the Stress

If students have reached grades 5 to 6 and are struggling to remember the multiplication facts, they may have built up some anxiety about them. Often it is worth spending time explaining how to read a 10 x 10 multiplication table and then having them systematically work through the table, crossing out those they know.

Everyone can draw a line through the "one times" and the "times one". Most know the doubles and the fives and tens. Quite a lot will know the squares. Spend some time on the commutative property. Students do not need to remember the word, but they should be able to say and understand why: *If I know 8 x 5 then I also know 5 x 8.* Show students how this reduces the number of facts to be remembered.

Remarkably, after removing the ones, twos, fives and tens and the squares, only 15 of the 100 facts remain. Emphasize that they are almost there! Most students will be able to cross off at least some of these 15.

Students can then make personal "prompt" cards for their remaining unknown facts. Have them set a personal target of, say, three to learn this week. During the week, help students work out how their three target facts relate to other facts they know. Periodically through the week, their partner should test them on the facts. When they have correctly recalled a fact, say, ten times, they cross it off their multiplication table list and store the card for later re-testing. Over

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the next several weeks, their partner should test them on previous weeks' facts, as well as their targeted three for this week.

Techniques for Mental Calculation

Students need practice with a wide range of strategies for calculating mentally. Mental arithmetic is flexible, purposeful and personal so it cannot be made routine. For example: to add 99 to 125, a sensible strategy would be to add 100 and subtract one; to add 64 to 125, you might add 60 and then four; to add 64 to 96, you might add the four and then the 60.

The fact that the calculation is done mentally does not mean that the presentation is always oral. Often we add a string of numbers that we can see as when playing Scrabble. When we add the digits in each column for column addition, the sums often go well beyond the basic facts and so mental arithmetic is needed. In some cases, students (like adults) will use some informal jottings on paper to help keep track of their thinking. Recording partial answers is widely used by adults and should not be discouraged. The choice is not between fully mental approaches and standard written approaches. The goal is flexibility and efficiency rather than standardization.

Students should use place value to extend the range of calculations they carry out mentally. For example:

- Count backwards and forwards in tens: 10, 20, 30, 40, ...
- Count in tens from any starting point: 14, 24, 34, 44, ... and 53, 43, 33, ...
- Add in tens, twenties and thirties, hundreds and so on, from any starting point: 23, 43, 63, ...
- Generalize basic facts: 8 + 7 = 15 so 18 + 7 = 25, 28 + 7 = 35; 6 x 7 = 42 so 60 x 7 = 420

The properties of the operations (when multiplying several numbers, the order does not matter), the relationships between them (division is the inverse of multiplication), number partitions and place value form the basis of the following mental calculation strategies.

Use relationships (commutativity and inverses)

- Adding: order does not matter. **4 + 27** is 27 + 4, so 28, 29, 30, 31.
- Multiplying: order does not matter. **24** twos is **2** twenty-fours, so 48.
- Subtracting: thinking of an addition might help. **13 – 8**, think "eight add what is 13?"
- Dividing: thinking of a multiplication might help. **63 ÷ 9**, think "how many nines make 63?"

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Compensate (partition and rearrange)

- Add: take some from one number to give to the other. **8 + 7** is 10 + 5; **68 + 37** is 70 + 35
- Multiply: take out a factor from one to give to the other. **15 x 6** is 15 times 2 times 3, so 30 times 3, so 90.
- Subtract: change the numbers by adding or subtracting the same amount. **62 – 37** is 65 – 40.
- Divide: change the numbers by multiplying or dividing by the same amount. **29 ÷ 5** is 58 ÷ 10.

Use compatible numbers and bridge

- Making change: **100 – 68**. Think "100 and what fits with 68" OR "It cost 68 cents. What's the change from $1?"
- Rearrange the order: **8 + 7 + 2** is 8 and 2 is 10 plus another 7 is 17; **68 + 27 + 12** is 68 and 12 is 80 plus 20 is 100 plus seven, so 107.
- Bridging: **9 + 4** is 9 + 1 + 3 = 10 + 3 = 13; **68 + 47** is 68 and 32 will make 100 and 15 left, so 115.

Front load (start with the biggest place)

- Bring on the tens and then the ones: **28 + 37** is 38, 48, 58 and 7 more, so 60, 65.
- Do both tens and then both ones: **68 + 37** is 90 + 15, so 100 and 5 more, so 105.

Imagine a number line

- Jump along or back: **364 – 198**: starting at 198, it takes 2 to get to 200 and another 164 to get to 364, so 166; OR starting at 364 go back 64 to 300, 100 more to 200 (so that's 164) and back 2 more to 198, so 166.

Multiply in parts (partition and multiply the parts)

- Round a number and adjust: **7 x 9** is 7 tens take away 7 ones, so 70 – 7, so 63; **99 x 6** is 600 take away six.
- Use place value partitions: **6 x 25** is 6 x 20 add 6 x 5, so 120 add 30.

Use factors

- Double, double, double: **4 x 14** is double double 14, so double 28, so 56.
- Change to a multiplication you know: **3 x 18** is 3 times 3 times 6, so 9 times 6, so 54.

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- Multiply by five: **5 x 8** is 8 fives, which is 4 groups of 2 fives, 4 tens so 40.
- Multiply by fifty: **50 x 72**; 50 is half of a hundred, so half of 72 hundred, so 36 hundred or 3600.
- Multiply by twenty-five: **36 x 25**; Notice the 25 and look for 4 to make 100, so 9 x 4 x 25 or 900.
- Doubling and halving: **45 x 14** is the same as 90 x 7, so 630.

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Computations:
Key Understandings Overview

Teachers will need to plan learning experiences that include and develop the following Key Understandings (KU). These Key Understandings underpin achievement of this family of concepts. The learning experiences should connect to students' current knowledge and understandings rather than to their grade level.

Key Understanding		Description
KU1	The same number fact will be true no matter how you count the objects or what the objects are.	page 118
KU2	We can think of a number as a sum or difference in different ways. We can rearrange the parts of an addition without changing the quantity.	page 126
KU3	We can think of a number as a multiplication or division in different ways. We can rearrange the factors of a multiplication without changing the quantity.	page 134
KU4	Place value and basic number facts together allow us to calculate with any whole or decimal numbers.	page 142
KU5	There are strategies we can practise to help us do calculations in our head.	page 152
KU6	There are some special calculating methods that we can use for calculations we find hard to do in our head.	page 164
KU7	We can calculate with fractions. Sometimes renaming fractions is helpful for this.	page 176
KU8	Rounding, imagining a number line, and using properties of numbers and operations help us to estimate calculations.	page 182
KU9	To use a calculator well, we need to enter and interpret the information correctly and know about its functions.	page 192
KU10	Thinking about what makes sense helps us to check and interpret the results of calculations.	page 202

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Grade Levels—Degree of Emphasis			Sample Learning Activities
K-3	3-5	5-8	
★ ★ ★	★ ★ ★	★ ★	K-Grade 3, page 120 Grades 3-5, page 122 Grades 5-8, page 124
★ ★ ★	★ ★ ★	★ ★	K-Grade 3, page 128 Grades 3-5, page 130 Grades 5-8, page 132
★	★ ★ ★	★ ★ ★	K-Grade 3, page 136 Grades 3-5, page 138 Grades 5-8, page 140
★	★ ★ ★	★ ★ ★	K-Grade 3, page 144 Grades 3-5, page 146 Grades 5-8, page 148
★	★ ★	★ ★ ★	K-Grade 3, page 154 Grades 3-5, page 156 Grades 5-8, page 159
★	★ ★	★ ★ ★	K-Grade 3, page 166 Grades 3-5, page 168 Grades 5-8, page 170
	★	★ ★ ★	Grades 3-5, page 178 Grades 5-8, page 180
★	★ ★	★ ★	K-Grade 3, page 184 Grades 3-5, page 185 Grades 5-8, page 187
★ ★	★ ★	★ ★	K-Grade 3, page 194 Grades 3-5, page 196 Grades 5-8, page 198
★	★ ★	★ ★	K-Grade 3, page 204 Grades 3-5,page 205 Grades 5-8, page 207

Key

★ ★ ★ **Major Focus**
The development of this Key Understanding is a major focus of planned activities.

★ ★ **Important Focus**
The development of this Key Understanding is an important focus of planned activities.

★ **Introduction, Consolidation or Extension**
Some activities may be planned to introduce this Key Understanding, to consolidate it, or to extend its application. The idea may also arise incidentally in conversations and routines that occur in the classroom.

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Key Understanding 1

The same number fact will be true no matter how you count the objects or what the objects are.

For calculation to make sense to students, they need to understand that numbers and operations have their own consistent meanings apart from any real-world situation. A student may count a collection of five pens and a collection of seven pens, put the pens together, count and conclude that there are 12 pens altogether. The question then arises whether it would still be 12 if counted another way. Or whether it is the same for pebbles? Or for people? Through experimentation the student has to be able to conclude that, no matter how you rearrange them, or from where you start your count, or what the items are, five items together with seven distinct items will (*must*) give 12 items.

Without this key understanding, the question "What is five add seven?" does not really make sense and nor does the statement "5 + 7 = 12". This idea, together with partitioning, is needed for students to see *why* when they add, they can count on from the largest number rather than counting the whole collection. The same idea underpins students' understanding of why the number facts for subtraction, division and multiplication can be relied upon—you do not have to work each one out for each new situation.

In the process of developing this key understanding, students should be assisted to use partitions and other strategies to help work out basic sums (to 10 + 10) and basic products (to 10 x 10), and record their findings in a variety of ways, leading to a systematic collection of facts using conventional notations. Having instant recall of basic number facts is helpful. This does **not** mean that students should learn them by rote—without understanding. Trying to remember things you do not understand increases the memory demands and makes learning more difficult for all students. This is particularly so for those who have intellectual disabilities, since poor memory is often a characteristic of their learning.

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We find it easier to remember and use things that we understand and which are well connected to other things. Students will remember number facts more easily if they understand what the facts are telling them, connect them to each other, and have the confidence and skill to work out those they do not recall by building upon those that they do. With suitable learning experiences that cluster basic facts into meaningful groups and reduce the memory load, most students can remember basic facts and should be helped to do so (See Background Notes, pages 108 to 112).

Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ use imagery and mental counting strategies to add and subtract small numbers, such as four and three, where the numbers are generated by a story ■ may not be confident that four plus three must always be seven
Quantifying	■ understand that the same basic addition or subtraction fact will be true regardless of how you count the collection or what the objects are ■ have constructed the addition facts to 10 + 10 and remember many of them
Partitioning	■ work with whole numbers independently of the particular context, including investigating patterns in the numbers themselves ■ remember the basic addition facts ■ have built up a table of multiplication facts to 10 × 10 and remember many of them
Factoring	■ remember almost all of the basic facts and can work out those they do not remember

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Sample Learning Activities

K-Grade 3: ★ ★ ★ Major Focus

How Many?

Have students use materials, such as pine cones or bottletops, to model an addition story involving change and then compare their answers. For example: Four butterflies are in your garden. If three more fly into your garden, how many will there be? Ask: Are the answers all the same? What if we counted them another way? Suppose the butterflies flew around? Ask students to count the total in different ways. Record in a picture and symbols.

Counting Chickens

Ask students to model stories. For example: Mother Hen gathered four of her chickens. If three more come back, how many will there be? Continue with different examples using the same numbers until students confidently claim it will always be seven. Ask those who claim this to justify to others. When students are convinced, ask them to say it in their own words and record as a number sentence: 4 + 3 = 7.

Imagining

Have students mentally add or take away two from a small collection of objects, such as five plastic animals. Ask students to imagine that they have taken away two animals. Ask: How many animals will be left? Repeat by adding three, four and five animals. Focus students on working it out by counting on; thinking of four as two and two and counting on by twos; and thinking of five as three and two.

Addition Table

Encourage students to build up their own table of addition facts, first to 5 + 5. Over time, build up the addition table to 6 + 6, then 7 + 7, and so on.

Number Cube Games

Play games such as this number cube game. Organize students into pairs. Give each pair three number cubes. Have students take turns to throw the cubes, add the numbers together and keep a running total on a calculator. During the game, ask: Which two numbers did you add together first? Why? At the end of the game (when one student reaches a total of, say, 50 or more), ask students to use number sentences to show their additions for at least one turn.

Doubles

Help students develop their repertoire of known facts by building on from the "doubles". For example: Have students use 6 + 6 = 12 to work out 6 + 7, 5 + 6 and so on.

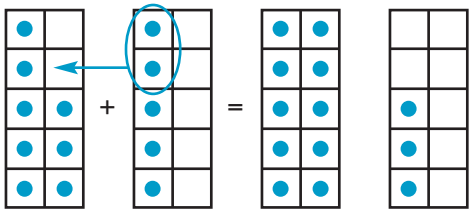
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Number Line

Extend the *Number Cube Games* activity on page 120. As students play the game, have them use a number line to check that their mental counting on and counting back always give the same result as counting, starting the count from one every time.

Compensating to Ten

Ask students to use two ten-frames (See Appendix: Line Master 6) to find ways of breaking up numbers to calculate. For example, to add 8 + 5, students move two from the five into the frame with eight to make ten and then add the remaining three. Extend so students can visualize the above movements.



8 + 5 = 13

Easy Calculations

Help students find an easy way of working out calculations, such as 7 + 4 or 5 + 7, using known combinations to ten. Ask them to share their strategies with the class. Extend later to include calculations with larger numbers by adding or subtracting from one of the numbers to make the other into a multiple of ten.

Double Collections

Have students double collections of materials, such as beans and counters, to make "twice as many" or "two times" and tell others the results. Begin with numbers from one up to four and ask students to extend the numbers themselves. Use a diagram to show the results and describe what the groupings mean, such as two sixes and six plus six. Use double and double again for students to work out four times a given collection.

Number Combinations

Ask students to recall number combinations from contexts where they made different partitions of the same number. For example: Think about when we made necklaces. We made one with eight beads. When five were red, how many were blue? Ask them to recall the situation and "see" in their mind's eye the parts of the collection.

Basic Facts to Ten

Help students to memorize basic facts to ten. Give each one a set of number facts cards. Ask students to work in pairs and take turns to put out three combination cards, such as 6 + 3, 9 + 0 and 3 + 4, two of which are combinations of the same number. If the partner can identify the odd card and say why, they can take the three cards. To vary the game, one student can set out two cards with equivalent totals for the partner to find another card to match the total.

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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Animal Patterns

Have students create an animal using Pattern Blocks, and then say how many blocks were used. For example: *My cow was made with eight triangles*. Ask: How many triangles would you need for five cows? Ask students who did not use materials to solve this and share with the class how they worked it out.

Same Numbers

Ask students to use their own strategies to solve and record solutions to multiplication problems involving the same numbers. For example: Jeremy can only carry seven plastic milk containers at a time to the recycling bin. How many does he take in four trips? Ask students to share the number sentences they used, say why they are the same and why the answers are the same.

Multiplication Facts

Help students to build up sets of related multiplication facts. For example: Ask students to draw one tricycle and say how many wheels then two tricycles and how many wheels, and so on. Encourage students to look for the pattern and say why five tricycles must have 15 wheels. Have students record the number sentences as the pictures are drawn to list the first five or six multiples of three.

Today's Number

Write a number on the board. Ask students to suggest calculations with that number as an answer. Record their calculations on the board. Ask: Are there any number sentences that belong together? Why? As students mention the four operations, build different groups. Ask: Can we rearrange the number sentences so they are in order? How can each set be extended?

Forgotten Facts

Ask students to explain to a partner how they could work out a fact they do not know or have forgotten. For example: *To find 6 x 5, I know it's 5 x 5 and another five*.

Doubles and Halves

Have students use doubles and halves to multiply and divide. For example: 4 x 7 is double double seven, which is double 14, or 28; 24 ÷ 4 is half of half of 24, which is half of 12, or six.

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Multiplication Doubles

Have students relate known multiplication "doubles" to the harder multiplication facts. For example: As a way of remembering 6 x 8, students could make a 6 x 6 grid and build on until they make 6 x 8. They work out tables they could put together to help work out the original, such as 6 x 6 and 2 x 6. Ask: Could you use addition instead of the 2 x 6? Which is easier?

Extending Doubles and Halves

Extend students' use of doubles and halves to find answers to multiplication, such as 8 x 12. For example: Halve eight and double 12 gives 4 x 24; halve four and double 24 gives 2 x 48; halve two and double 48 gives 1 x 96.

Concentration

Have students use addition and multiplication examples to construct cards to play games such as Concentration. Pairs of cards are made by putting together different representations of the same number. For example: 3 x 2 and 2 + 2 + 2 would be a matching pair.

Constant Calculations

Have students use the constant function on the calculator to find multiples. For example: When learning the four times table, press 0 + 4 then = = = to find the multiples of four. Ask students to predict what will be next and then to verify their prediction. Ask: Why can't a number with seven in the ones column be a multiple of four?

Grid Patterns

Have students make a multiplication grid by placing numbers along the top and down the side and the answers within the grid. (See right.) Look for patterns within the grid. Ask: Why are the numbers above the diagonal the same as below? How can this help find answers to tables you don't know?

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Looking for Patterns

Have students investigate patterns in the answers of times tables. For example: In the nine times table, the digits of each answer add to nine, the numbers in the ones column go up by one and the numbers in the tens column go down by one.

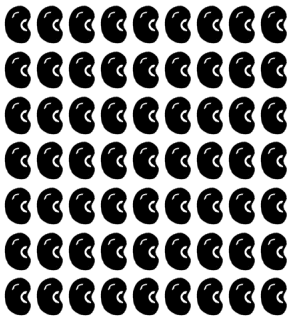
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Sample Learning Activities

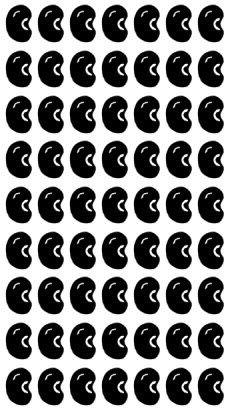
Grades 5-8: ★ ★ Important Focus

Array Facts

Have students cut out arrays to represent different multiplication facts. Ask them to take turns to show an array and write facts about that array. For example: 7 x 9, 9 x 7, 63 ÷ 9 = 7, 63 ÷ 7 = 9, $\frac{1}{9}$ x 63 = 7, $\frac{1}{7}$ x 63 = 9.



7 x 9 = 63
63 ÷ 9 = 7



9 x 7 = 63
63 ÷ 7 = 9

Fewer Facts

Encourage students to use turn-arounds to reduce the number of facts to be learned. Present stories such as: Jessica knows her ones, twos, fives and ten times tables. How can she use what she knows to work out 5 x 8? Ask students to show on a 10 x 10 grid how using turn-arounds reduces the number of facts to be learned.

Unknown Facts

Ask students to use known facts to work out unknown facts. For example: How does knowing 9 + 6 help you work out 8 + 6? How does knowing 9 x 6 help you work out 8 x 6? What was different?

Difficult Facts

Have pairs of students use factors to work out the more difficult basic multiplication facts. For example: To work out 7 x 8, you can say 7 x 4 x 2 or go further and say 7 x 2 x 2 x 2. Ask students to sort a range of multiplication facts according to whether using factors helps.

Equivalent Sentences

Have students use equivalent number sentences to make calculating easier. Pose this problem: Katy needs to work out 56 ÷ 8. She knows the multiplication facts well but is not as sure of the division facts. Ask: How could Katy use multiplication to help her solve the division sentence?

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Checking Calculations

Encourage students to experiment with their calculators as a tool to help them practise basic facts. For example: To practise the eight times table, key in 8 x 2. Now key in 3 =, 5 = and 1 =. Ask: Is your calculator still multiplying each number? Try other numbers, saying the answer before pressing the equals key. Students can experiment with programming different "times tables" into their calculator.

Choice Bingo

Invite students to play *Choice Bingo*. This is similar to the normal bingo game except that students make up their own 4 x 4 boards by writing their choice of numbers on the grid. The teacher holds up multiplication facts from 5 x 5 to 10 x 10 on flash cards. The first student to cross off a whole row, column or diagonal is the winner. Ask: Is 37 a good number to use? Why? Ask students to decide which numbers are best for their grid.

42	32	56	27
60	36	50	49
64	90	72	45
48	54	81	70

Constant Quantities

Invite students to use a calculator to explore what happens in multiplication when you double one number and halve the other. Ask: Why doesn't this change the quantity? How does knowing about the effect of doubling and halving help you to work out more difficult multiplication facts?

Square Numbers

Have students use square numbers as known facts to work out unknown facts. For example: Ask students to construct a multiplication grid (See *Grid Patterns*, page 123) and locate the pattern of square numbers. Shade in the facts that could be easily worked out from a square number. For instance, if you know 7 x 7, you can work out 6 x 7 by saying 49 take away seven. You can also work out 8 x 7 by adding seven to 49.

Number Facts

Invite students to use a range of strategies to give calculations that match a given number. For example: A small group of students select the number cards one to nine from a pack of cards. They each start with the same number of cards. The first player turns over two cards and arranges them to make a two-digit number. The next player then states a multiplication or division fact about the number. This continues until the last player who can state a fact keeps the pairs of cards. The person with the most cards at the end of the game is the winner. If a prime number appears, the cards go to the bottom of the pack. Later, primes can be included to encourage the use of fractions.

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Key Understanding 2

We can think of a number as a sum or difference in different ways. We can rearrange the parts of an addition without changing the quantity.

Partitioning into part-part-whole is the basis of students' understanding of the *meaning* of addition and subtraction, and the relationships between them, and of why addition is commutative but subtraction is not. This idea is also included here because it underpins flexibility in calculation with both whole and fractional numbers.

Students should develop the idea that they can partition collections and objects into part-part-whole without changing the total quantity, and that they can often do this in different ways. They should experiment by moving quantities from one part into the other to discover that:

- the quantity does not change when the objects are rearranged
- there are patterns linking pairs of numbers

Thus, they should learn to think flexibly of numbers as the sum and difference of other numbers. This is the basis of skilful calculation, because it enables students to see, for example, *why*:

- to find 8 + 274 you *must* get the same result if you start at 274 and count on eight as if you start with eight and count forward 274
- to find 274 + 8 you can think of eight as 6 + 2, use the six to "fill up" the seventies to get 280 and then add the two
- to find $\frac{4}{5} + \frac{3}{5}$ you can mentally "shift" one-fifth from the $\frac{3}{5}$ to the $\frac{4}{5}$ to make up a whole (one), which you can then combine with the remaining two-fifths, giving $1\frac{2}{5}$

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Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ understand that it makes sense to rearrange the parts of an addition for whole numbers ■ can use a known basic fact to work out facts they do not yet know, such as eight plus nine must be double eight plus one ■ use basic known addition facts to work out subtraction facts ■ use informal written strategies based on partitioning to add and subtract two-digit numbers
Partitioning	■ can use informal mental strategies based on partitioning to add and subtract two-digit numbers ■ partition a number into an addition to assist in their multiplication and division *For example*: A student may think of 27 × 4 as four 20s plus 4 sevens.
Factoring	■ mentally partition a number into an addition to assist in their multiplication and division
Operating	■ partition fractions to add and subtract them

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Sample Learning Activities

K-Grade 3: ★ ★ ★ Major Focus

Same Amounts

Have students move objects to develop the idea that the same amount can be partitioned into two groups in different ways without changing the quantity. For example: Six students sat down to eat pieces of fruit. At first three had bananas and three had apples. Then two had bananas and four had apples.

Moving Items

Ask students to model stories about the movement of items from one group to another, such as *Ten in the Bed* (Dale, 1999). Ask students to describe each combination of those in and out of bed. There are ten altogether and three are in the bed; how many are on the floor? If you're the only one left out of ten in the bed, how many are on the floor? Students record the combinations.

Hands

Ask students working in pairs to use fingers to show numbers between six and ten as five plus something, that is, show six as , eight as . Have one student hold up his or her hands and hide them then the partner say the number shown. Practise until students recognize one to ten immediately.

More Hands

With students working in pairs, ask one student to hold up some fingers on each hand. The partner then imagines some fingers from one hand moving over to fill the other hand, and says how many in all. For example,  and  makes .

Tosses

Have students take a handful of counters that are red on one side and white on the other. They record how many in all (say, eight). They then toss the counters and record how many of each colour is showing. Continue and make a record of combinations. Ask: What stays the same (eight in all)? What changes (how many each of red and white)? Have you got all the possible combinations? How do you know?

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Record Symbols
Extend the previous *Tosses* activity by recording in symbols. For example: Four yellow beads and five red beads make a necklace of nine beads, 4 + 5 = 9.

Number Combinations
Represent partitions in activities, such as *Tosses*, on page 128, and *Record Symbols*, above. For example: Provide beads in two colours and ask students to make a necklace of nine beads, then represent it using spots or drawings on cards. They then sort their cards into the same number groups, putting together the cards with four yellow and five red. Make a label for each combination under students' direction. Draw out that all necklaces have the same number of beads but in various number combinations of yellow and red.

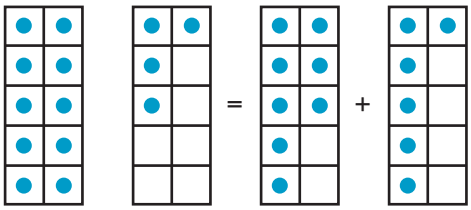
Reverse Order
Extend the previous activities to draw out that we can reverse the order of the numbers in an addition. For example: Have students rearrange the nine beads on their necklace so that all of one colour is at one end and the other colour at the other end. They hold the necklace and read the number sentence from left to right: 4 + 5. Their facing partner looks at the same beads and reads the number sentence also from left to right: 5 + 4. Then draw out that these are the same.

Three Groups
Extend the previous activities to partitioning into three groups.

Favourite Number
Have students make, describe and display a poster about their favourite number, showing the various number combinations. For example: Eight is four and four. It is five fingers on one hand and three on the other. It is four pairs of shoes.

One-to-Ten Partitions
Ask students to show partitions for numbers from one to ten. For example: Make cards with spots in two clusters. Write the total number on the back of cards and use them in games, such as Snap and Concentration, to develop recognition of partitions.

Combinations to 20
Have students use two ten-frames to investigate ways of representing combinations to 20. For example: 14 = 8 + 6, 7 + 7 = 14, 20 – 6 = 14 and so on. Ask: How many different combinations are possible for 14? How do you know?



14 = 8 + 6

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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Marble Bags

Have students start with 16 marbles in one bag and work out how many different ways they can put the marbles into two bags. Ask them to record the combinations and look for patterns in the number sentences to help find other possibilities. Extend by asking: If there were 19 in the bag, would your pattern still be helpful?

Number Combinations

Use a grid or number line to construct the combinations for a given number. Ask: How do you know you have found all possibilities?

Patterns

Have students drop materials, such as beads, blocks and counters, onto a card divided into three sections, and record the different combinations for each number up to 18. Ask them to organize the number sentences into a pattern to see if they have found all possible combinations. Ask: Is 7 + 6 + 5 the same as 6 + 5 + 7 or 5 + 6 + 7?

Number Cube Patterns

Repeat *Patterns*, above, using three number cubes.

Card Games

Have students play card games, such as Snap or Fish, to make combinations to a predetermined number, rather than pairs. For example: In Fish, they may have four in their hand and need to make 11, so they ask for seven.

The Answer Is ...

Ask students to make up an addition or subtraction that has a specified answer, such as 30. Ask them to make up as many as they can. Help them to put their sums (or differences) in order and fill in any gaps. For example:

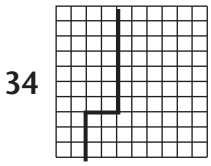
1 + 29 = 30 2 + 28 = 30 3 + 27 = 30.

Ask: What happens to the second number as the first goes up by ones? Challenge them: I worked out that 237 + 492 = 729. Make up some other pairs that add to 729, without doing the calculation. If I increase the 237 to 238, what do I need to do to the 492? Repeat for subtractions.

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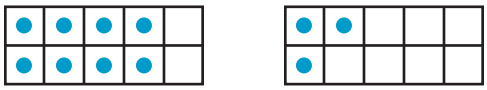
Grid Partitions

Have pairs of students find and record partitions of 100 using a 10 x 10 array on grid paper (See Appendix: Line Master 7). To play this game, students choose the left or right side of the grid to score. Together they toss two, ten-sided number cubes (See Appendix: Line Master 8). One cube represents tens, the other represents ones. Use the numbers from each turn to partition a 100 square. Students count the tens and ones to find the other number and record each combination of 100. Students add their scores as they go. The first to reach (say) 500 is the winner.



Ten-Frames

Encourage students to develop a visual image of number partitions to 20 by saying how many counters there are in ten-frames as they are flashed on an overhead projector. Ask students to explain how they were able to see "that many" on the frame. Extend to two frames or to 100 frames (10 x 10, 4 x 25, and so on).



Hide the Blocks

Play *Hide the Blocks* in pairs, using a given number of blocks. Player One closes their eyes while Player Two hides some blocks under a container. Player One then works out how many are hidden, using the beginning number and how many are left outside the container. Have students record the number sentences as they go. Extend to two- and three-digit numbers by using material bundled in tens and 100s.

Groups of Tens

Invite students to use materials grouped in tens, such as linking cubes, popsicle sticks and Base Ten Blocks, to construct as many representations of a given number as possible. For example, 37 could be represented as three tens and seven ones, two tens and 17 ones, one ten and 27 ones, or 37 ones. Have students record their representations and justify each by showing how their groups of materials link to the representations. Extend to include three-digit numbers.

Breaking Up

Have students use materials grouped in tens, such as linking cubes, popsicle sticks and Base Ten Blocks, to show how they might break up a number to calculate. For example: 86 + 47, shown as 86 + 4 to make 90 and then add 43. Have students record the breaking of numbers and movement from one number to the next with a written explanation or with numbers and signs.

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Sample Learning Activities

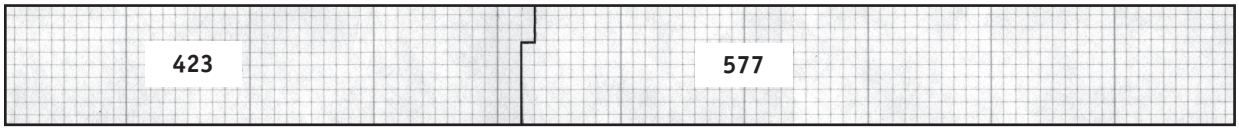
Grades 5-8: ★ ★ Important Focus

Grid Partitions

Have students identify partitions of 100 by using 2-mm grid paper (See Appendix: Line Master 7) and drawing a line to partition a 100 square into two parts. (See *Grid Partitions*, page 131.) Ask: Where would you draw the line so that your partner can say how many in each part at a glance? Using a new 100 square each time, ask students to take turns to quickly recognize the two parts of 100. Have students share strategies that make this recognition easy. Repeat the activity but imagine the 100 square is $1.

Thousand Grid

Extend the *Grid Partitions* activity above using a 1000 grid (See Appendix: Line Master 9). Ask students to say how they could see at a glance how many hundreds, tens and ones there are in each of the two parts. Ask: How do the strategies differ from the previous activity?

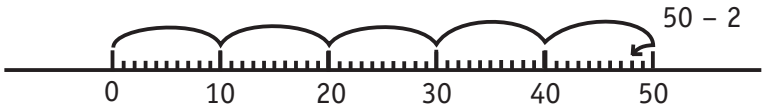


Small Change

Have students partition money to avoid small change. For example: Li's takeout food came to $10.25. He gave the cashier a $20 bill and 25 cents. Why did he give the extra 25 cents?

Number Line Leaps

Ask students to use place value to partition numbers. For example: When showing 48 on a number line, take the least amount of one-unit or ten-unit jumps to get to 48. For instance, jumping five ten-jumps and back two one-jumps is quicker than jumping four ten-jumps and eight one-jumps. Ask students to represent the different jumps as an addition or subtraction to make 48.



Further Leaps

Extend *Number Line Leaps*, above, to numbers requiring jumps of 100, 1000 or 0.1.

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Pass the Number

Play *Pass the Number*, where students use partitioning to represent a number in different ways. Any number, including decimals and common fractions, is written at the top of a piece of paper and the paper is passed around the class. Each student writes an alternative form of the number, using either an addition or subtraction. Students try to circulate the paper without any combination being repeated.

Talk About It

Invite students to share their conjectures about partitioning to make calculating easier. Sue said: *You can add the same number to both numbers in a subtraction problem without changing the calculation*. So, 47 – 16 is the same as 50 – 19 or 51 – 20. Ask students to decide if this is true and talk about how knowing this could be useful to calculate examples such as 78 – 29 and 96 – 17. Ask: Does this conjecture only work for subtraction?

Rewriting Number Sentences

Have students partition numbers and rewrite a number sentence in a variety of ways that do not change the result. For example: 73 – 38 can be written as: 73 – 25 – 13; 73 – 21 – 17; (60 – 30) + (13 – 8); 73 – 30 – 8. Ask students to exchange number sentences with a partner and check that the total has not changed. Ask: Which one of these ways would be most useful when calculating?

Rewriting Addition

Have students partition numbers in an addition number sentence to make calculating easier. Ask: How can 36 + 28 be rewritten in a different way without changing the result? Encourage students to share ways that the parts can be rearranged and check that the total stays the same. Ask: Which arrangement is the easiest for you to calculate this particular example?

Uncommon Numbers

Repeat the Rewriting Addition activity above with uncommon numbers, such as: 1499 + 1501; 3000 – 1499; 15.01 + 14.99; 15.01 + 1.499; 1.49 + $1\frac{1}{2}$.

Large Numbers

Ask students to partition large numbers, fractions and decimals in different ways. For example: The answer is 14.25. What could the addition question be? What could the subtraction question be?

Compatible Numbers

Encourage students to recognize pairs of numbers that combine to make one, 100, 1000 or a decimal fraction, such as 0.1. Ask students to generate pairs of numbers that add to make a given number, such as 1000, and randomly record the pairs on a piece of paper. Have them exchange sheets with a partner and join up all the compatible numbers. Ask: What clues did you use to know how many hundreds to look for? What about the tens and ones?

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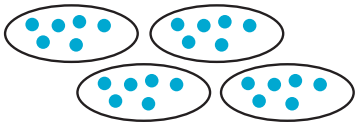
Key Understanding 3

We can think of a number as a multiplication or division in different ways. We can rearrange the factors of a multiplication without changing the quantity.

Partitioning quantities into equal groups is the basis for students' early understanding of the *meaning* of multiplication and division and the relationships between them and of fractions. This idea underpins flexibility in calculation with both whole and fractional numbers.

Students should develop the idea that a number can be decomposed and recombined into its factors in a number of ways without changing the total quantity involved. They can use this to write numbers as products (multiplication) or quotients (division). They should experiment by arranging and rearranging actual collections and objects into equal groups and arrays, skip counting either orally or along a number line or with a calculator, and exploring the numbers themselves.

For example:



24 is four groups of six

24 = 4 x 6

24 ÷ 4 = 6

24 ÷ 6 = 4



24 is three rows of eight

24 = 3 x 8

24 is eight columns of three

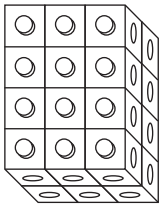
24 = 8 x 3

24 ÷ 8 = 3

24 ÷ 3 = 8



24 = 12 x 2



Numbers can also be written as the product of a number of factors. Thus, 24 is 2 x 3 x 4, which can be represented with a two by three by four rectangular solid made with cubes. As well, 24 is 1 x 2 x 2 x 6, although this would be difficult to show with materials so we usually rely upon explorations with the numbers themselves.

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Being able to think flexibly of numbers as the product of two or more factors is helpful when calculating. It enables students to see that to calculate 24 x 5 we could think of the 24 as 12 x 2, so that 24 x 5 must be the same as (12 x 2) x 5. We can then mentally change the order of the multiplications to 12 x (2 x 5) which is 12 x 10, or 120. Alternatively, we could think of five as 10 ÷ 2 or $\frac{10}{2}$, then multiply 24 by ten and divide by two—a common "trick of the trade". Students should not be taught such "tricks" mindlessly but rather should be helped to develop the flexibility with numbers that leads to such strategies.

Links to the Phases

Phase	Students who are through this phase. . .
Partitioning	■ understand that it makes sense to rewrite whole numbers as factors and can do it for numbers for which they know the basic relevant facts, such as 20 or 36 ■ can further break down numbers such as 24 and 36 into three or four factors and realize you can do multiplications in any order and get the same result ■ draw on the knowledge that you can do multiplications in any order and get the same result and on known multiplication facts to produce new facts *For example*: Since four sixes are 24, eight sixes must be two groups of four sixes, or 48; four 60s is four groups of six tens, which is 24 tens or 240.
Factoring	■ can rewrite larger whole numbers into factors and even into prime factors ■ understand *why* you can rearrange the factors of a number and do the multiplication in any order ■ freely draw upon this understanding for their mental calculations

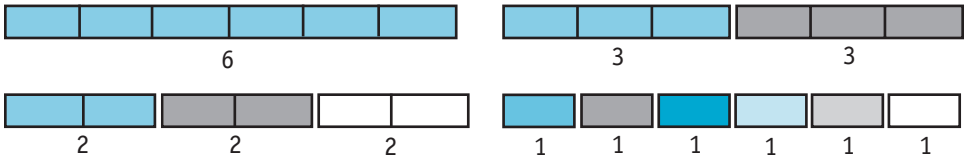
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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

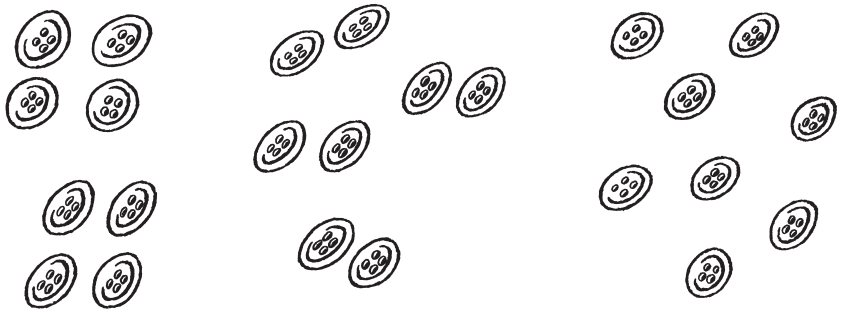
Equal Groups

Have students work with materials that are fixed representations of numbers, such as Base Ten Blocks or linking cubes, using different colours to show equal groups. Ask students to investigate and record the different equal groups that can be made for each given number, such as six.



Arranging Objects

Ask students to arrange and rearrange a collection of, say, eight objects into equal groups. Ask: What equal groups can be made from eight things? Ask students to record in their own way using drawings, diagrams or numbers on individual cards and display on a chart about "eight". Ask: What about nine things?



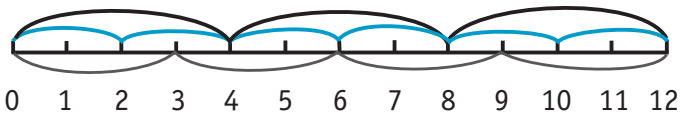
Sharing Cookies

Have students draw diagrams to show grouping in stories, such as *The Doorbell Rang* (Hutchins, 1987). Ask them to share the 12 cookies into two equal groups and write a matching number sentence, such as 2 x 6 = 12. Repeat as each visitor in the book arrives.

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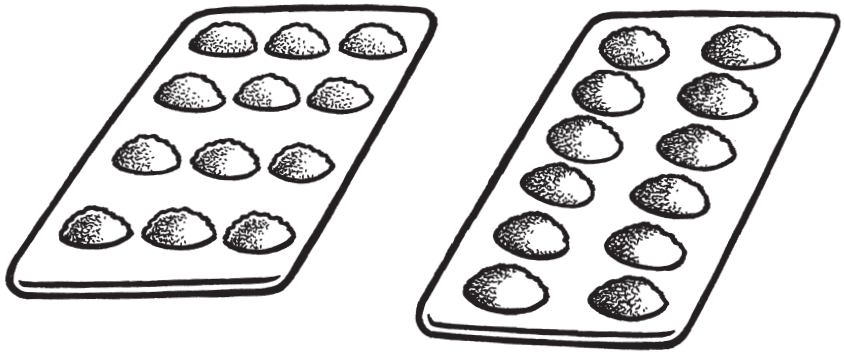
Number Line

Ask students to skip count to a given number along a number line, using groupings to see that the count always reaches that number. For example: 12 has equal groups of one, two, three, four and six. Draw out that 3 x 4 and 4 x 3 are different partitions of the same collection.



A Different Point of View

Have students investigate arrays from different positions and describe the groupings to see that rearrangement does not change the quantity. For example: A baker had many different trays for making 12 muffins. Ask students to draw, on grid paper, all the trays the baker could have, and describe their different arrays, such as three rows of four, six rows of two. They can cut them out and sort them to see that some are the same shape—three by four and four by three. Ask: How do they look different on the paper? Why? Are they both still 12?



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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Arrays

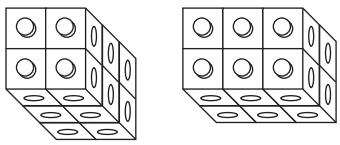
Help students develop a visual image of tables to 20. Briefly show an array of dots, such as 2 x 4. Ask students to say how many across, how many down and how many altogether. Turn the array 90 degrees and ask again. Ask: How is this array the same (different) from the last array? Have students write a sentence for each, such as 2 x 4 and 4 x 2 are both eight.

Making Arrays

Have students construct all arrays for the number 24 using grid paper, pegboards or blocks. Ask them to record each using a multiplication sentence, such as 6 x 4 = 24, and a division sentence, such as 24 ÷ 6 = 4. Ask: What do each of the numbers show in the array? Why isn't there a sentence using the number five or seven? Draw out that you cannot make groups of five or seven from 24 blocks. They are not factors of 24. Ask students to write a list of factors for 24.

Prisms

Ask students to build rectangular prisms from a set number of blocks. Ask them to record each as they go by writing number sentences, such as 2 x 2 x 3 = 12. Ask: Is 2 x 2 x 3 equal to 2 x 3 x 2? Can you make a different shape for each? Why is it that some numbers are used (one, two, three, four, six and 12) and some are not (five, seven, eight, nine, ten, 11)?



Planning

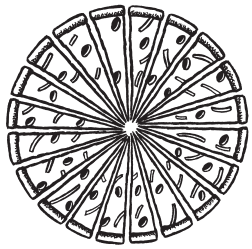
Have students partition equipment for games, such as cards and marbles, into all possible equal groups to find out how many groups of students can play at one time. Ask them to list the groupings and keep these with the equipment as instructions, such as 12 cards can be shared equally between two, three, four, six and 12 players.

More Planning

Extend *Planning*, above, to involve larger numbers in each collection (48 straws, 52 playing cards, 100 dominoes, 100 marbles). Ask students to predict which numbers of groups will "work" for the specified collection size. Check by skip counting on a calculator to reach the total number of the collection.

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Multiple Pieces



Extend the previous *Planning* activities to include those involving partitioning a whole into parts. For example: I am not sure whether there will be four or five of us to share the pizza. How many pieces should I cut it into to make sure that either way, we can all take the same number of pieces? Draw out the mathematical idea that the number of pieces has to be a multiple of both four and five.

Tree Planting

Ask students to find a number divisible by two other numbers. For example: Dad bought enough maple saplings for three children to plant equal numbers of saplings. Then a friend arrived, but luckily the number of saplings we had could still be shared evenly. What is the smallest number Dad could have bought? Draw out that the number has to be a multiple of both three and four. Ask: Are there other numbers of saplings that would work?

Using Factors

Have students use factors to solve problems. For example: We have 24 tomato seedlings to plant in the garden. How could we arrange them into rectangle shapes? How do you know you have found all possible arrangements? How can you work out factors of numbers without using material?

Finding Multiples

Invite students to use the constant function on the calculator to find multiples of a number and predict whether a given number will be a multiple. For example: Will 51 be a multiple of four? This can become a game by taking turns to predict the next number before pressing the =.

Using Shapes

Have students use shapes to investigate factors. For example: Give students a random handful of straws and ask them to decide how many triangles they can make from the collection. How many rectangles? Pentagons? Record using pictures and number sentences. Draw out the idea of factors by asking: Why do you get straws left over for some shapes and not for others?

Multiples of Three

Build up multiples using the inverse of the *Using Shapes* activity above. Have students make one triangle using three straws, two using six straws, and so on. Ask them to draw their triangles. Then have them write the list of multiples of three next to their diagrams to show how many straws were used altogether.

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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Pass the Number

Write a number, say 32, at the top of a piece of paper and pass it around the class. Ask each student to write an alternative form of the number on the paper, using either a multiplication or division, with no combination being repeated. Encourage students to include fractions and decimal fraction examples. Ask them to justify interesting examples, such as 0.25 x 128 and 2^5.

x	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Turn-arounds

Ask students to explain scenarios such as: Jodie knows her one, two, five and ten times tables. How does knowing about turn-arounds help her work out 5 x 8? Show on a 10 x 10 tables grid how this reduces the facts to be learned.

Frog Hops

Have students partition quantities into equal-sized fractional amounts. Start by making different fraction number lines. Use them to solve frog-hopping problems. The frog can only make hops of the same length. How many ways can the frog hop to $1\frac{1}{2}$? How many ways can the frog hop to 2? How many ways can the frog hop to $\frac{3}{4}$? Ask students to write a number sentence for each kind of frog hop. For example: 6 x $\frac{1}{4}$, or 3 x $\frac{1}{2} = 1\frac{1}{2}$. Ask: Can you use your number line partitioned into thirds to end up at $1\frac{1}{2}$? Why?

Multiple Slices

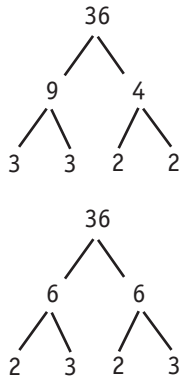
Have students decide how many parts they need to partition things into so that they can be shared equally by groups of different sizes. For example, say: Grandfather likes to cut the cake so that it can be shared equally by the grandchildren, but he isn't always sure how many will come. Ask: If sometimes three come and sometimes five come, what number of pieces could he cut? What is the smallest number of pieces? Repeat for numbers that have factors in common, such as six and eight (so that 48 works but is not the smallest). Repeat for other pairs of numbers that have no factors in common. Draw out that the number of pieces has to be a multiple of each number of children.

Factor Bingo

Ask students to draw a 5 x 5 grid and fill the grid with numbers. Hold up a number, such as 36, and have students cross off any numbers on their grids that

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multiply to make 36. The student who is the first to cross off a line or a diagonal is the winner. Discuss the most suitable numbers to use in the grids and the strategy for being able to cross off the most numbers. Ask: Why would 36 be a better number to hold up than 37?



Making Factor Trees

Invite students to investigate factors and products by making different factor trees for the number 36.

Comparing Factor Trees

Have students compare the different factor trees for the same number. Ask: What is it about the numbers in the last row that tells you that you have all the factors?

Divisibility

Ask students to generate multiples of numbers, such as two, three, four, five, six and nine, and investigate divisibility rules of each number. Ask: What numbers will divide evenly into 2004? How do you know?

Divisibility Rules

A whole number is divisible by:

2 if the number is even

3 if the sum of the digits is divisible by 3

4 if the number represented by the tens and ones digits is divisible by 4

5 if the ones digit is 0 or 5

6 if the number is divisible by 2 and by 3

8 if the number represented by the hundreds, tens and ones digits is divisible by 8

9 if the sum of the digits is divisible by 9

10 if the ones digit is 0

Doubling Strategy

Have students look for multiplications where factorizing enables them to use a doubling strategy. For example: 19 x 4 can be solved by saying double 19 is 38 and double again is 76. Ask: Would you be able to use doubling to solve 34 x 2, 45 x 4, 8 x 15 or 16 x 34? Why?

Strategies

Extend the *Doubling Strategy* activity, above, by having students build up a chart of other strategies, using factors to multiply. For example: To calculate 36 x 25, you can say 9 x 4 x 25 which is 9 x 100. Ask students to use factors to work out 28 x 25 and compare this way to the way an adult they know works it out. Ask: Which way is easier? Why?

Easier Multiplication

Have students factorize numbers to multiply more easily. Introduce the activity by displaying the following example on an overhead projector (See Appendix: Line Master 10):

- 8 x 32 can be rewritten as 2 x 2 x 2 x 32. That's double 32, three times.
- 14 x 15 can be rewritten as 7 x 2 x 3 x 5. That's the same as 5 x 2 x 7 x 3, which is 10 x 21.

Multiplying by Ten

Ask students to look for multiplications where factors of two and five enable them to multiply by ten. Introduce the activity by displaying the following example on an overhead projector (See Appendix: Line Master 11): 15 x 18 is the same as (3 x 5) x (3 x 6), which is the same as (3 x 5) x (3 x 3 x 2), which is the same as 3 x 3 x 3 x 10.

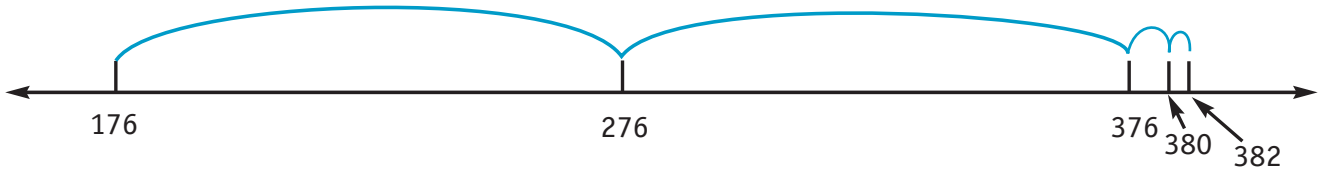
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Key Understanding 4

Place value and basic number facts together allow us to calculate with any whole or decimal numbers.

From the earliest years, students should be challenged, through practical problem situations, to calculate with numbers larger than those that they readily do by counting or using known facts. As suggested in Key Understandings 2 and 3, partitioning is the key to calculation. By decomposing and composing numbers into convenient forms, we can turn any required calculation into one involving a series of applications of basic facts.

In most cases, computational strategies rely on the flexible use of place value, where numbers are partitioned in both standard (47 = 40 + 7) and non-standard (47 = 20 + 27 or 47 = 43 + 4) ways. Although numbers can be partitioned in many ways, partitioning involving tens, hundreds, thousands and so on is particularly helpful, because these numbers are easy to calculate with. For example, when a student finds 97 + 26 by shifting three across from the 26 to the 97 to make 100 + 23, it is because she or he knows it is easier to add 23 to 100. When a student finds 176 + 206 by jumping along a number line in hundreds first, she or he is also using place value.



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The place-value concepts in Whole and Decimal Numbers, Key Understanding 6 (See *Number Sense*, page 72), should be extended and deepened as students experiment with ways of computing with larger numbers using manipulative materials, diagrams, calculators, visual images and patterns in the numbers themselves. Teaching place-value concepts separately as a prerequisite to calculation is unnecessary and likely to be ineffective. Rather, students should construct increasingly deeper concepts of part-part-whole and place value as we challenge them with opportunities for making estimates and computations.

Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ partition into tens and units to add and subtract two-digit whole numbers ■ may use mental or informal paper-and-pencil methods including diagrams or jottings
Partitioning	■ explain their written or mental method for adding and subtracting in a way that shows they understand why they work ■ can use standard place-value partitions to help them multiply and divide a whole number by a single-digit whole number
Factoring	■ have the flexibility to partition whole numbers in both standard and non-standard ways to meet the particular demands of the current computational task
Operating	■ have extended this flexibility to the partitioning of decimal numbers

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Hands

Have pairs of students each hold up fingers to show a number between five and ten, such as six as  and eight as . To calculate 6 + 8, they put the two fives together to show ten and add to the others to give four more. Practise until students are able to quickly say the sums.

Groups of Ten

Use groups of ten to count large collections. For example: Give students a handful of materials, such as beans, buttons or toothpicks. Ask them to arrange these into groups of ten to count by tens then ones, or skip count by twos or threes.

Breaking Up

Encourage students to talk about how they "break up" numbers to make them easy to work with. Look for responses that show use of basic facts such as: *I know five and three is eight; I thought of five and three as four and four*. With 5 + 4, focus on answers such as *four and four and one more*.

Adding Ten

Suggest students use a number line and calculator to add ten to any number. Have students write 3 + 10 = 13 and 13 + 10 = 23. Ask: Which of the digits in 23 will change when you add ten? Why? What happens when we add ten again? What if you take away ten? What if you add nine? Take away nine? Continue this for one to ten.

Partitioning

Have students build on *Adding Ten*, above, and partition numbers to help calculate. For example:

- "five and something", so with five and eight you know eight is five and three more. So it's 5 + 5 is ten and three more
- expand this later into "ten and something"
- think of nine as "one less than ten", so think of 9 + 8 as "ten and eight less one"
- eight is "two less than ten", so 8 + 7 is "17 minus two"
- seven and six could be "double six add one"

Hundreds

Extend *Partitioning*, above, to numbers into the hundreds for students to count on and back through the decades by tens. For example:

- 160 – 30: 30 is three tens so it's 160, 150, 140, 130
- 140 + 50: 140, 150, 160, 170, 180, 190

Students can count how many tens by keeping track on their fingers.

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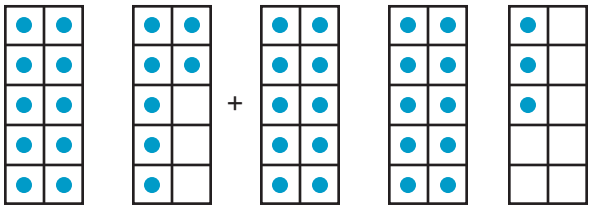
Class Collections

Store class sets of glue, scissors, thick markers and so on in rows of ten, and have a student check by glancing how many have been collected or need to be collected. Ask: How many tens can you see? Can you combine some to make another group of ten?

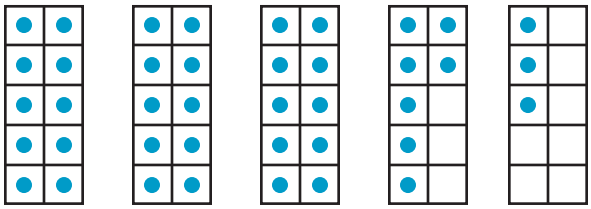
Combining Tens

Have students use ten-frames (See Appendix: Line Master 6) to experiment with ways to calculate with larger numbers. For example: Ask students to set up dots on ten-frames to show 17 + 23. Ask: How does moving your frames to put the tens together help? How can you put the singles together to more easily see what the total is? How can we use numbers and signs to show what you did with the tens and ones?

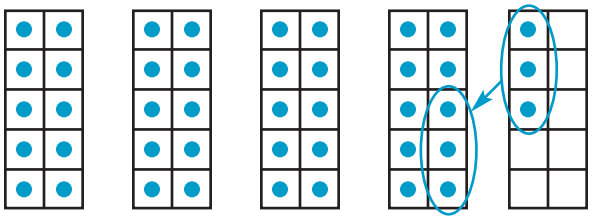
17 + 23 = ?



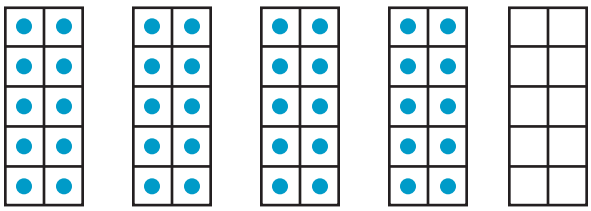
Move the frames to put the tens together:



Move the singles together:



17 + 23 = 40



Orders

Use materials such as paper clips or notebooks—in singles, packs of ten or 100—for solving individual students' orders. For example: At the driving range, golf balls can be obtained as singles or packs of ten or 100. Jack wants 72. What packs will be in his order? Mae Lin wants 142. What are the different ways we could make up her order?

Larger Orders

Expand on the *Orders* activity, above, and ask students to suggest other combinations for filling large orders by using diagrams or referring to the numbers alone, apart from materials, to work out these combinations.

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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Building on Basics

Challenge students to build from their basic facts by saying: If I know 3 + 4 = 7 then I also know ... because ... Ask: Does knowing that 3 + 4 = 7 help you know what 30 and 40 is? Have students make personal lists of what they know from the one fact and add to this as they discover more. This activity can be extended later to include multiples, such as 3 x 4 and 30 x 4, and large whole numbers and decimals, such as 0.3 + 0.4 = 0.7.

Problems in Context

Have students visualize or represent problems in a way that enables them to use place value to calculate an answer. For example: Trisha's family has to travel 123 km to their campsite. They have already gone 87 km. How much further is there to go? Students can jump forward along a number line, or visualize the same saying: *87, 90 (that's three) 100, 110, 120, (that's 30) and 123 (three more) so that's 36*.

Combinations to 100

Encourage students to solve problems by using combinations to 100. For example: Lawson had $87 and Filomena had $123. How much more does Lawson need to earn to have the same amount? 100 – 87 is 13 so 13 + 23 is 36.

Math Methods

Present an operation horizontally on the board, such as 62 – 23. Allow time for students to calculate in their head and then ask them to explain what they did. Record methods on the board and draw out how most methods break up the numbers. Ask: Why did you break the numbers up in that way? Why did you put those two numbers together first?

Non-standard Place Value

Have students solve problems using non-standard place value partitions by moving from one number to another on a 100-chart (See Appendix: Line Master 20). For example: We need 74 balloons for a party and we have 36. How many more do we need? This could be solved by counting on by tens: 36, 46, 56, 66, 76, – 2. That's 40 – 2, so the answer is 38. Or counting back from 74: 74, 64, 54, 44, 34, + 2. That's 40 – 2 = 38. Ask: Which do you find easiest? Why? What other methods could be used?

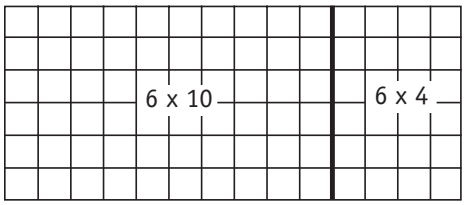
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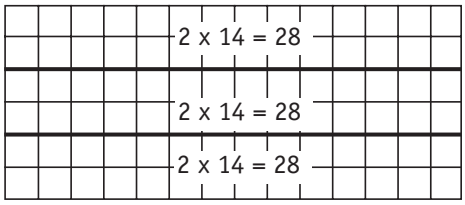
100-Chart

Ask students to visualize movements on a 100-chart to solve problems involving addition and subtraction. Discuss how moving up or down a row is the same as adding or subtracting ten and moving back or forth along a row is the same as adding or subtracting one.

Grid Partitions

Have students explore ways of breaking up numbers for multiplication calculations using 2-mm grid paper (See Appendix: Line Master 7). For example: Represent 6 x 14 and find an easy way of breaking up the grid to help work out the total.





Ask students to share the various partitions and decide which ones make calculating easier. (See Case Study 1, page 150.)

Moving Squares

Have students draw representations of three-digit numbers on ten-squared grid paper (See Appendix: Line Master 12) to help add or subtract. When adding, ask students to look for ways of "moving" some squares from one number to the other to make groups of tens. For example: 136 + 248, move two from 136 and add two 248 to make 250.

Bundling Materials

Ask students why it is helpful to bundle materials into tens to solve division problems. For example: When sharing 75 counters among three students, making bags of ten to begin with means you can share out using groups rather than ones.

Personal Strategies

Have students use their own methods to calculate, such as strategies like those in the Bundling Materials activity above. Ask them to show how they have worked it out by writing the way they carried out the calculation, and why they chose to write down those particular numbers. They could do this on an overhead transparency.

Explaining Procedure

Show the students a procedure used by "another class" to solve a calculation. For example: 273 x 4 solved as 800 + 280 + 12 = 1092. Ask: What did they do with 273 x 4 to get 800/280/12? How did it help them to solve this problem? Ask students to explain what is helpful about this procedure, and how they would modify it for their own use. Have them use it for similar and other ways of calculating, such as 4 x 250 add 4 x 23.

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

100-Chart

Have students add and subtract numbers by visualizing a 100-chart. Show them a 100-chart for a few minutes and then remove it from view. Ask: What number is below 43? How do you know? What number is three to the right of 72? How do you know? You are at 34, go right two steps and up three. Where are you now? You are at 68. How do you get to 75? Ask students to describe the jumps needed to calculate 24 + 39 and 83 – 47.

1000-Chart

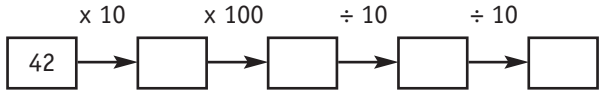
Extend *100-Chart*, above, by having students make up a 1000-chart with one to 100 along the top row, 101 to 200 on the second row and so on. Ask similar questions as students use the chart to work out and explain their jumps.

Leap Along a Number Line

Have students make jumps of one, ten or 100 on a number line to calculate 423 + ☐ = 632 or 891 – 674 = ☐.

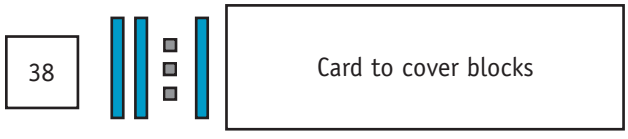
Multiples of Ten

Invite students to build up their own rules for multiplying and dividing by multiples of ten. For example, they could make up number trails for their partner, who can work backwards to check their answer.



Sequences

Have students add on different sequences of ones, tens, 100s or 1000s to a starting number. For example: Write 38 on a card and line up Base Ten Blocks covered with a long card.

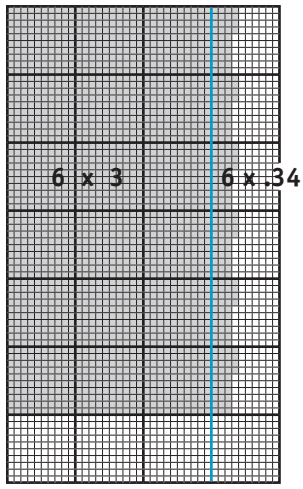


Students draw back the card to the right to sequentially reveal the blocks as their partner counts on from the number: 48, 58, 61, 71. This can be extended by including hundreds (flats) or thousands (big cubes).

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Decimals

Have students use 2-mm grid paper (See Appendix: Line Master 7) with a 10 x 10 square representing one whole to calculate with decimals. For example: For 6 x 3.34 a student might draw around the squares showing six groups of 3.34 and show that 6 x 3 = 18 and write 18; 6 x three-tenths is one and eight-tenths, that's 1.8; 6 x four-hundredths is two-tenths and four-hundredths, that's 0.24. Add to reach a total. Discuss the different ways students used the grid to work it out.



Partitioning Numbers

Organize pairs of students to partition numbers to help do calculations. For example: For 99 x 27, they might see 99 as 100 – 1 and think *That's a hundred 27s less one 27*, then jot down: 2700 – 27
2600 + (100 – 27)

To calculate 4 x 27, they might think *Four groups of 20 and four groups of seven*, and jot down the partial products on paper: 80 + 28.

Later, for 34 x 27 they might think *That's 30 27s, add four 27s*, leading to something like the standard algorithm.

Rewriting Number Sentences

Have students partition numbers and rewrite a number sentence in a variety of ways. Introduce the activity by displaying the following example on an overhead projector: 73 – 38 can be written as 73 – 13 – 25; 73 – 21 – 17; (60 – 30) + (13 – 8); 73 – 30 – 8. Ask students to exchange number sentences with a partner who checks that the total has not changed. Ask: Which one of these ways would be most useful when calculating?

Rewriting Multiplication

Repeat, *Rewriting Number Sentences*, above, for multiplication. Introduce the activity by displaying the following example on an overhead projector: 55 x 67 can be written as:
50 x 67 + 5 x 67; 100 x 67 ÷ 2 + 10 x 67 ÷ 2; 50 x 60 + 50 x 7 + 5 x 60 + 5 x 7.

Division Number Sentences

Have students use purposeful "rough work" to calculate division number sentences, such as 272 ÷ 16. For example, a student might record:

10 160
5 80
= 240 , 32 left, that's another two 16s, so 17.

Ask students to share and try a different strategy. Ask: Which was easiest? Why?

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CASE STUDY 1

Sample Learning Activity: Grades 3-5—Grid Partitions, page 147

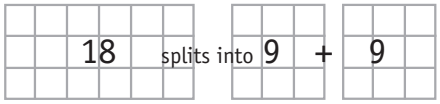
Key Understanding 2: We can think of a number as a sum or difference in different ways. We can rearrange the parts of an addition without changing the quantity.

Key Understanding 4: Place value and basic number facts together allow us to calculate with any whole or decimal numbers.

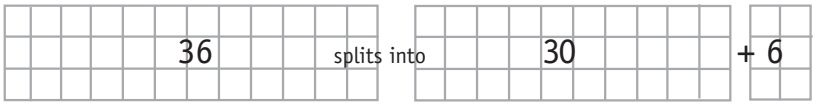
Working Towards: The end of the Quantifying and Partitioning phases

TEACHER'S PURPOSE

Students in Mr. Romero's grade 3/4 class had used tiles and grid paper to explore partitioning.



18 is 6 columns of 3 = 3 columns of 3 + 3 columns of 3



36 is 12 columns of 3 = 10 columns of 3 + 2 columns of 3

Mr. Romero wanted to introduce the idea that splitting quantities into tens often makes calculating easier. He decided to give the students several problems over the next few weeks that would give him the opportunity to draw this out.

Mr. Romero asked students to plan how many dog biscuits they needed to make treat bags for the dogs at a local shelter. The numbers involved made it hard to count.

ACTION

Mark used an array representation to help in the task. He was working out 12 bags with eight dog biscuits in each bag. He drew eight dots in a row, then started to do another row of eight below it. Mark stopped, then continued down the page drawing the first dot only in a total of 12 rows. He explained that he did not have to draw all the biscuits. He could keep counting the first row over and over for all the rows. He wrote the total for each row as he went.

CONNECTION AND CHALLENGE

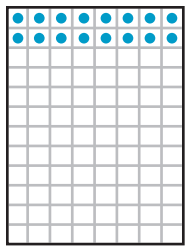
Mr. Romero used the overhead projector to show Mark's approach to the class on a grid array. Mr. Romero asked Mark to draw the dog biscuits from his first

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bag in a row, with one biscuit in each grid square, and then the next bag in a row beneath that. Mr. Romero then challenged him: Without drawing any more biscuits, make a rectangle around all the squares you'll need to finish drawing all your bags of dog biscuits.

Mark drew a 12 x 8 rectangle and said: *It's got eight squares for eight biscuits on that row, and that's one bag. Then I counted 12 rows down so there's 12 bags.*

Mr. Romero asked the class if Mark needed to draw in all the biscuits to find out how many. Everyone said no, he could just count the squares. Mr. Romero then asked them to show their treats and bags on grid paper. All but one or two understood how the rows of squares showed how many dog biscuits for each bag.

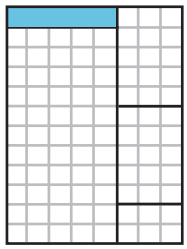


Mark's rectangular array for 8 dog biscuits in 12 bags.

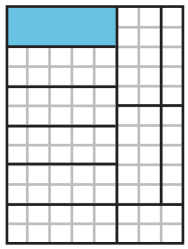
DRAWING OUT THE MATHEMATICAL IDEA

Mr. Romero went back to Mark's grid and talked about how his array could help him calculate how many dog biscuits he needed in total.

Thus, Mr. Romero moved from Mark's original array to partitioning in tens with its link to place value, which Su Li related to the work the class had done with tiles.

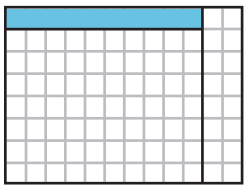


1. Mark: *I did it in 5s, so you can do 5, 10, 15 ... 90 and then it's six more.*



2. Mr. Romero: *Great! That was easier than 1, 2, 3, ..., Is there a way to use 10s to do it?*

Mark: *You could put the 5s together like this and do 10, 20, 30 ..., 80, than 85, 90, and 6, 96, that's faster.*



3. Mr. Romero: *Yes! Now if we start again with it turned this way, can anyone find an even easier way to use 10s to help work it out?*

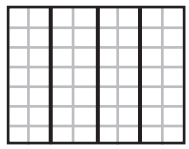
Su Li: *I'm thinking about the tiles and you could make 12 across into 10 and 2 then it's 8 rows of 10, that's 80 and 8 is 88 and another 8 ... that's 2 to make 90 and 6 left, 96.*

If students are to believe that counting the squares tells them how many dog biscuits, a connection has to be made between the quantities shown on the grid and the quantities in the problem situation. This is not as obvious to students as we sometimes assume.

OPPORTUNITY TO LEARN

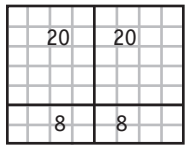
The students then returned to their own problems and showed the required quantities on an array while Mr. Romero worked with individuals. Jason immediately split his 14 x 3 array into a ten by three and four by three. He then counted in tens, 10, 20, 30 and added on the 12 he knew to get 42.

Sandy was working with a much smaller multiplier (7 x 8) and she did not know the basic fact. Nevertheless, she used the idea of partitioning and first split her 7 x 8 array into two seven by fours, then each of those into two seven by twos. She wrote 14 + 14 + 14 + 14, then began doubling. 14 + 14 is 20, 24, 25, 26, 27, 28 (counting the last on four fingers). So 28 + 28... Mr. Romero helped her see how this calculation was represented on her array by the first split. They then drew a line to separate each 28 into 20 and eight and she immediately saw how to complete the calculation.



14 + 14 + 14 + 14

28 + 28



Sandy calculated by saying 20, 30, 40, 48 (then counted on by 2s, pointing at the squares) 50, 52, 54, 56.

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Key Understanding 5

There are strategies we can practise to help us do calculations in our head.

Students should learn to see mental arithmetic as the first resort when they need to calculate. Mental computation refers to doing calculations "in your head" and involves much more than recall of basic facts. It is not necessarily quicker than written computation. Sometimes it will be slower. Its importance lies in its portability and flexibility.

If students develop the idea that "mental math" should be quick or obvious, they may be reluctant to try to do calculations in their head.

Mental methods are usually different from written methods and students rarely get better at mental computation from practising written computation. The best mental methods help us to store small parts of the calculation in our heads as we go along. Thus, many people find it easier to do 37 + 26 mentally by adding the tens first (47, 57) and then the ones (and six more is) than to mentally add the ones, regroup and then add the tens as we do for the written algorithm. It helps to work from left to right so that the parts of the number are mentally stored in the order they are written and said—starting with the biggest part and progressively getting closer to the answer as you gather the sub-calculations: 37, 47, 57, 60, 63.

As they proceed through elementary school, students should develop an expanding repertoire of mental strategies to assist in calculating and estimating mentally—initially with whole numbers and later with money and simple fractions. A list of strategies for mental calculation is provided in the Background Notes on pages 113 to 115. Students should be encouraged to develop personal mental strategies, to experiment with and compare strategies used by others, and to choose strategies to suit their own strengths and the situation. This requires a *high* level of number sense

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based on a strong grasp of place value and of the relationships between operations. In general, mental methods backed up by informal written records should receive more attention than standard written methods.

Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ mentally add and subtract small numbers generated from stories; that is, they work with small, easily visualized numbers that they count in their "mind's eye"
Quantifying	■ have begun to use some mental strategies to work out addition and subtraction facts that they do not know, and to make simple extensions
Partitioning	■ use mental strategies to work out basic multiplication facts that they do not yet remember ■ have begun to build up a repertoire of strategies, such as compensation, which help them add and subtract "easy" two-digit numbers mentally
Factoring	■ have considerably extended their repertoire of strategies for reducing memory load ■ can mentally add and subtract two-digit numbers and mentally multiply and divide by single-digit numbers and multiples of ten or "easy" numbers, such as 4×32 ■ will try mental computation first for one-time calculations
Operating	■ are skilled in mental computation with whole numbers and money and calculate mentally with easily visualized fractions

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Counting On

Have students say which number they are able to count on from when adding small collections, such as 3 + 2. Think of two, now count on three more. Think of three, now count on two more. Think of four and count on three more. Try thinking of five and counting on three more. Ask: What other numbers could you think of, then add three more on? Does it always give the same result as counting them all?

Counting On and Back

When students have begun to use "counting on", ask: Why did you start from three? Does it give you the same result as starting the counting at one? Compare "counting the group" with "counting on" or "back" when combining and separating small collections using materials and diagrams. Extend this later by asking: Does it give the same result when you count on from the bigger number when adding, say, eight and five? Does it help?

Adding Chunks

Begin a counting sequence at one. Have students take turns to count aloud the next ten, using their fingers to keep check. After several students have counted, ask students to predict where the next student will stop. Ask: Can you think of another way to find the next group of ten without counting by ones? Repeat later with each student counting the next five.

More Chunks

Extend *Adding Chunks*, above, by starting at any number to nine. For example: Start at three and use fingers to count aloud to 13 then move to another student to count ten more. After several students have counted, ask students to predict where the next student will stop. Ask: Can you think of another way to find the next group of ten without counting by ones? Later, link to numbers in a column on a 100-chart and extend, adding into higher numbers.

Number Line

Have students use the class number line as an aid to hold numbers that are too large for them to keep in their head when mentally calculating. Ask them to think of the number line in their head and imagine counting on or back four from a given number to help them build a mental number line for themselves. For example: There are four students away today. Use your mental number line to count back to see how many are present.

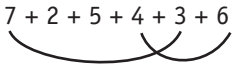
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Skip Counting

Vary the previous *Number Line* activity to skip count up and down a number line. Teach someone else how to do it.

Compatible Numbers

When adding a list of numbers, ask students to look for numbers they can combine more easily. Focus on adding the numbers that combine to make ten first, then go back to add others. For example:



Ask: Which numbers add together to make ten? Is this way of adding a string of numbers more helpful than adding in the order they are written? Why?

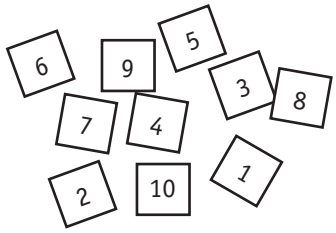
Did You Know?

Place a scattered collection of ten or so "number tiles" (See Appendix: Line Master 13) on the table in front of students and ask them to add to find the total score. Observe how they go about it.

Students in grades 2 to 4 were asked to do this. Many were able to add the numbers, using and building on basic facts. However, almost all added the numbers as they came to them. For example: 6 + 9 = 15 + 5 = 20 + 8 = 28 + 7 = 35 + 4, and so on. Very few thought to use known combinations to ten to make the task easier.

Thus, although the students remembered basic facts and were quite good at adding on small numbers, they could not use them flexibly to make the task easier and probably more accurate. Indeed, in this case, they could have easily physically moved the tiles together to make combinations of ten, but rarely did.

It is important to encourage students to use their number knowledge in flexible ways to make calculation easier. This is an essential aspect of number sense.



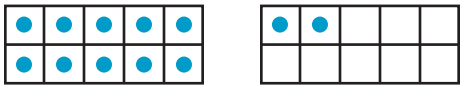
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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Compatible Numbers

Encourage students to develop a visual image of number partitions to 20 by using counters in one or two ten-frames (See Appendix: Line Master 6) as they are flashed on an overhead projector. Have students say how many counters and how many empty spaces they saw and record these combinations to 20.



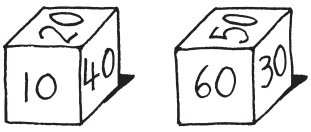
What Is the Question?

Give students a number, such as 36, and give them one minute to write as many questions as they can which have that number as the answer.

Choose the Operation 1

Give each pair of students a number cube. Ask each student to write down 50, and take turns to throw the cube and decide whether to add, subtract, multiply or divide the number on the cube to his or her own score (starting with 50). The winner is the first to reach zero or 100.

Choose the Operation 2



Extend *Choose the Operation*, above, to include two number cubes and allow a combination of operations. Extend again to practise calculating with tens by labelling a cube with ten through to 60 (See Appendix: Line Master 14). Start with 500 and have a goal of zero or 1000.

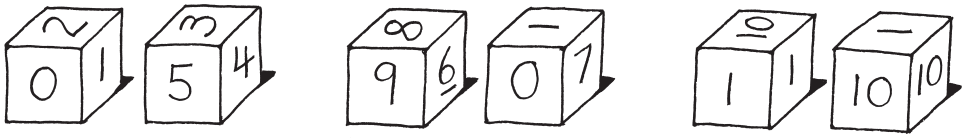
Target Practice

Make three number cubes with these numbers on their faces (See Appendix: Line Master 15):

Cube 1: 0, 1, 2, 3, 4, 5 Cube 2: 6, 7, 8, 9, 1, 0 Cube 3: 10, 10, 10, 1, 1, 1

One student in each group picks a target number less than 100 and throws the cubes. Using the numbers shown on the cubes, each group member writes a number sentence that gets as close as possible to the target. Any operations can be used. The winner is the one coming closest to target.

Cube 1 Cube 2 Cube 3



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Roll the Number Cubes

Use the number cubes from the previous activity, *Target Practice*. Break the class into two teams and roll the cubes. Students in each team have three minutes to write as many correct number sentences, such as 2 x 8 + 10 = 26, as they can. When three minutes is up, members of one team read out their sentences. If both teams have the sentence, they both cross it off. Any sentence not crossed off is a point for that team. Continue until one team reaches a designated winning score.

Decade

Remove the face cards from a deck of cards and let the ace be one. Have players choose a target decade, such as the thirties. Each player draws four cards and uses any operation to come up with a number sentence to make a number in the target decade. All four cards must be used. A point is scored for each correct sentence.

Wipe Out

Ask students to enter a two-digit number on their calculator. They then use addition to wipe out—replace with a zero—the first two digits from right to left to get 100. Do another, and another, and another. Ask: Can you do it in just one step?

Doubling

Have students use doubling to add. For example: 39 + 47 could be changed to 30, add 40, add eight and eight. Give students a series of additions, such as 36 + 55, 18 + 49, 67 + 88, and ask: What double would be helpful for each?

Multiply the Parts

Ask students to investigate how numbers can be broken up to make multiplying easier. For example: 12 x 8 can be thought of as 10 x 8 add 2 x 8; 99 x 6 can be thought of as (100 x 6) – 6. Ask students to partition the numbers in a problem, solve it mentally and share how they stored it as they went along.

Compensation

Give students an example of another student's thinking to try. For example: Abbey found the sum 58 + 37 by taking two from the 37 and adding it to 58 to make 60. Then she added 60 to 35 to get 95. Have students try Abbey's strategy of moving parts from one to another for: 19 + 21, 22 + 34, 39 + 12. Ask: Why did you move that part of the number? What compatible numbers did you make?

Front-end Loading 1

Have students visualize counting on and back on a number line or 100-chart to solve an addition or subtraction problem. For example: 24 + 37; start from the tens; repeat, starting from ones. Ask: What did you say to yourself when you started from the tens? What did you say when you started from the ones? Which way is easier to keep track of the answer? Draw out that it is easier to remember the parts in our head if we do the tens first because we keep saying the new number as we go, such as 37, 47, 57 and four more is 61.

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Grades 3-5: ★ ★ Important Focus

Front-end Loading 2

Have students mentally calculate 42 + 37, then share what they said to themselves. Record their thoughts on the chalkboard. Draw out the front-end strategies where:

- you start at the large number then bring in the tens then the ones of the other number, such as 42 add 30 that's 72, then add seven, that's 79
- you do the tens first then the ones, such as 40 add 30, that makes 70, add two, equals 72, add seven equals 79

Have students use the front-end strategy to add and subtract. Later, extend to three-digit calculations, such as 234 + 121 or 664 – 231.

Front-end Loading 3

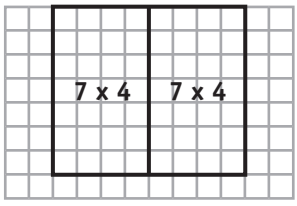
Repeat *Front-end-Loading*, above, to draw out subtraction strategies. For the example, 66 – 28:

- 66 take away 20 is 66, 56, 46, take away eight, that's 46, take away six (40) and take away two (38)
- 60 take away 20 that's 40, then add the six that's 46, take away six, that's 40, take away 2 that's 38.

How Did You Do It?

Have students choose a strategy for doing calculations in their head, then report on what they did and why they chose that approach. Draw out how certain approaches help you remember the bits (See Case Study 2, page 162).

Constructing Arrays



7 x 4 x 2 = 56
7 x 8 = 56

Ask students to find factors by constructing all possible arrays for a given number using grid paper, pegboards or blocks. Write a list of factors for each number and use these to compute the more difficult basic multiplication facts. For example: To work out 7 x 8, you can say 7 x 4 x 2 or go further and say 7 x 2 x 2 x 2. Have students sort a range of multiplication facts according to whether using factors helps or not.

Factoring

Have students explore how factoring numbers in a multiplication number sentence can make mental calculations much easier. For example: 4 x 16 can be rewritten as (2 x 2) x 16, which is double 16, and double again. Ask students to solve similar calculations where using factors of two enables students to use the doubling strategy, such as 6 x 32, 8 x 21 and so on. Draw out the fact that you can store the numbers in your head as you go, so there is not as much to remember.

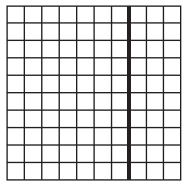
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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

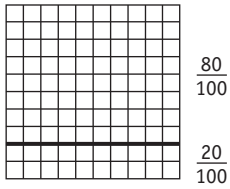
Compatible Numbers 1

Have students practise recognizing partitions of a 100 square on grid paper at a glance (See Appendix: Line Master 7). Ask students to draw a line to partition a 100 square into two parts. Show your diagram to a partner and see how quickly they can recognize and name the two parts. Repeat the activity using ten adjacent 100 squares to partition 1000 into two parts (See Appendix Line Master 9).



Compatible Numbers 2

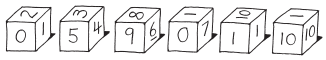
Extend *Compatible Numbers*, above, by making the 100 square a unit of one. Have students name the two parts as compatible decimal or common fractions and percentages that add to make one, such as 0.3, $\frac{30}{100}$, 30% with 0.7, $\frac{70}{100}$ and 70%.



$\frac{20}{100} + \frac{80}{100} = \frac{100}{100} = 1$

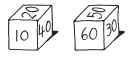
Target Practice

Play *Target Practice,* on page 156, and then extend it to use more number cubes or to include fractions and decimals on the faces.



Choose the Operation

Play *Choose the Operation*, on page 156, and extend it to practise calculating with tens by labelling a number cube ten through to 60 (See Appendix: Line Master 14). Start with 500 and have a goal of zero or 1000. Extend again by rolling the cube ten times in front of the whole class. Ask each student to decide whether to multiply by one, ten or 100 and record the result. After ten rolls they total their scores and the person closest to 1000 wins.



Compensate

Have students partition numbers in a calculation to make use of compatible numbers. For example, say: In 88 + 47 you can subtract 12 from 47, put it with the 88 to make 100 and you're left with 35, that's 135. 75 – 27 is 75 – 25 – 2, which is 48. Ask: How can you split the numbers in these calculations to use that strategy: 88 + 32, 25 + 78, 100 – 47.

Wipe Out

Play *Wipe Out*, on page 157. Have students begin with a three-digit number and initially wipe out digits from right to left to get to 1000. Ask: Can you wipe out the three digits in fewer steps?

Number Sentences

Ask students to write number sentences that could be mentally solved using compatible numbers, such as 45 + ☐ = 100, 23 + 77, 1000 – 625. Swap with a partner to solve.

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Grades 5-8: ★ ★ ★ Major Focus

Front-end Loading 1

Have students visualize a number line and make jumps of one, ten or 100 to solve problems, such as 45 + ☐ = 107, 423 + ☐ = 632 or 891 – 674 = ☐. Ask students to say how they jumped to their goal number and kept track of the number parts as they went. Ask: Is it easier to first add or subtract the ones, or the largest place in a number?

Front-end Loading 2

Present what two students said when they were calculating 465 + 132. For example: Ben said *465, 565, 595, 597*; Jeremy said *500, 590, 597*. Ask students to work out what they were thinking, describe the difference between the strategies and choose one to calculate these: 335 + 523, 464 – 212. Ask: Would you use the same approach to add 453 + 175 in your head?

Front-end Loading 3

Ask students to use this approach to mentally solve a range of two-digit multiplications. For example: 34 x 12 could be thought of as 34 x 10 add 34 x 2. Ask: Are all two-digit multiplications made easy using this approach? Which ones were? Which ones weren't?

Factoring

Have students factor numbers in a multiplication number sentence to calculate mentally. For example:

- 8 x 32 can be thought of as double, double, double 32
- 14 x 15 is written as 7 x 2 x 3 x 5. That's ten times 21, which is 210.

Give students a range of multiplication calculations to solve by factoring.

Doubling

Invite students to look at a range of two-digit multiplications where factoring enables them to use a doubling strategy. For example: 19 x 4 can be done by saying double 19 is 38 and double again is 76. Ask: Would you be able to use doubling to solve 34 x 2, 45 x 4, 8 x 15 and 16 x 34? Why?

Factors of Two and Five

Repeat *Doubling*, above, looking for the factors of two and five, which enable you to multiply by ten. For example: 15 x 18 is the same as 3 x 5 x 3 x 3 x 2, which is the same as 3 x 3 x 3 x 10, which is 270.

Renaming Fractions

Have students decide when it is useful to rename fractions as decimals and vice versa to make calculation easier. Ask them to decide if renaming helps in the following examples if only an approximate answer is required: 20 600 000 x 0.52; 0.34 x 621; 824 x 0.26; 20% of $42. Students can write calculations with decimals where renaming would help and share these with a partner.

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Multiply the Parts

Display Line Master 16 (See Appendix) to present the following approach for multiplying 99 x 6: 100 x 6 is 600, take away the extra six is 594. Have students use the same strategy to work out 99 x 4; 99 x 8; 7 x 99. Ask: Would this same strategy work with 19 x 3? How?

Strategies

Extend *Multiply the Parts*, above, by making a chart of other strategies using factors to multiply mentally. For example: To calculate 36 x 25, you can say 9 x 4 x 25, which is 9 x 100. Ask: What is it about the strategies that make it easier to do in your head?

Compensating with Fractions

Have students use the compensate strategy above when adding and subtracting fractions, such as fractional quantities in a recipe. To calculate $\frac{2}{3} + \frac{2}{3}$, you might say $\frac{2}{3}$ add $\frac{1}{3}$ is one. Add the other $\frac{1}{3}$ is $1\frac{1}{3}$. To calculate $3\frac{1}{3} - 1\frac{2}{3}$ you might say $3\frac{1}{3} - 1\frac{1}{3}$ is two, take the other $\frac{1}{3}$ leaves you with $1\frac{2}{3}$. Have students work out other similar examples using this strategy.

More Compensating with Fractions

Pose this problem: James rewrote the addition $\frac{3}{5} + \frac{4}{5}$ as $\frac{1}{5} + \frac{2}{5} + \frac{4}{5}$. Ask: Why would he do that? How does it make it easier to work out? Try this method on similar examples.

Partitioning with Fractions

Have students partition to solve problems involving fractions of whole amounts. For example:

- One-quarter of the 24 buttons were red. How many was that?
- Only three-quarters of the class of 32 remembered their bathing suits on the first day of swimming lessons. How many students was that?
- The sale is advertising 20% off all jeans. If the pair of jeans that James wants cost $50, what will the new price be?

Sale Time

Have students calculate the savings made during a "15% off" sale. Students can talk and write about what 15% actually means and then mentally calculate 15% of the cost of a backpack worth $35. Some students might think *15 cents for every dollar, so that's 15 x 35 cents*. Others may say *10% is $3.50, add half of that, which is $1.75, which gives you $5.25 off*. Students could use flyers from supermarkets and other stores to mentally calculate common discounts, such as 50%, 25% or 33% off.

Converting

Ask students to decide when it is useful to convert between fractions, decimals and percentages to make calculating easier. Ask: Does converting help in the following examples: 20 600 x 0.5; 0.5 x 620; 800 x 0.25; 20% of $40; 50% of 200.

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CASE STUDY 2

Sample Learning Activity: Grades 3-5—How Did You Do It?, page 158

Key Understanding 5: There are strategies we can practise to help us do calculations in our head.

Working Towards: The end of the Quantifying and Partitioning phases

TEACHER'S PURPOSE

Ms. Chung was working with her grade 4 class, over a number of weeks, identifying the range of strategies the students used to calculate mentally. She planned to heighten the students' awareness of how some mental strategies make it easier to store the sub-totals as you go along, hence reducing memory load.

ACTION AND REFLECTION

There is not one way of solving an example like this. Many different ways are possible. The idea is to encourage students to see the different ways and then to choose a method that best suits them. This lesson begins the journey towards understanding that there are many different ways of solving this sort of problem.

Ms. Chung wrote 38 + 27 across the board horizontally and asked the students to solve the problem in their head. After some thinking time, the majority of the students offered 65 as the answer, and she wrote this on the board. Ms. Chung asked the students to tell each other what they had done.

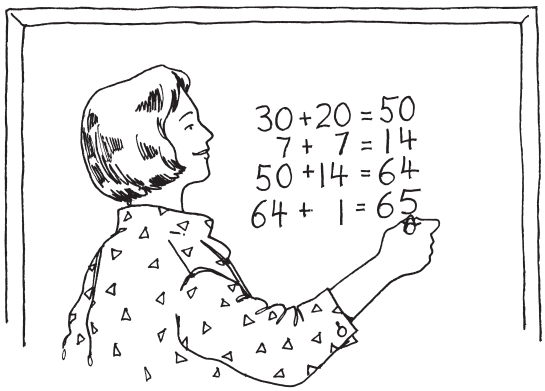
Hayley: *I did 30 add 20 that makes 50, and then I did 7 + 7, that makes 50 + 14, that's 64 and then one more, that's 65.*

Ms. Chung wrote on the board: 30 + 20 = 50; 7 + 7 = 14; 50 + 14 = 64; 64 + 1 = 65

Then Ms. Chung asked Hayley why she had done 30 + 20 first.

Hayley: *It makes it easier if I start from that side. I can remember where I'm up to.*

Ms. Chung: *So… Hayley says that by adding the tens first it's easier to keep track of where she is up to in her head, easier for her to remember. Did anyone else start from the tens first?*



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DRAWING OUT THE MATHEMATICAL IDEA

Many of the students had started from the tens side. They agreed that it was easy to remember what they had worked out this way, but Ms. Chung wanted them to think about this even further. She pointed to the initial number sentence 38 + 27 and asked them to say the first numbers aloud.

Ms. Chung: *What part do you say first?*

Students: *30.*

Ms. Chung: *So which number do you think about first?*

Students: *30.*

Ms. Chung: *So what number do you think about first with 27?*

Students: *20, just like it is in reading it.*

James: *I tried to start with 38 but it was too hard, so I changed it to 40. Then I could say 40 add 27. I can do it really quick. 40 add 27 makes 67.*

As he spoke, Ms. Chung wrote on the board: 40 + 27 = 67 to provide a focus for the students' thinking.

James: *I did 67 – 2, that makes 65.*

Ms. Chung wrote this up as 67 – 2 = 65 then asked: *So who can tell us why James took the two away from the 67?*

Sacha: *Because he had added two to the 38 in the beginning.*

Ms. Chung emphasized this point: *Yes, so because he had changed the 38 to the 40 in the beginning by adding two, he had to keep this in his head and then remember to take away the two at the end.*

Hayley helped out at this point and told the class that in the way she had worked hers out, she had to remember to add one at the end.

Hayley: *Because I couldn't do eight and seven, I did 7 + 7 because that's easy and then I had to remember to add the extra one afterwards. 7 + 7 is easy, it's a double like 5 + 5 and 6 + 6. You just know it, and then you don't have to stop and work it out like I have to do with 7 + 8.*

Ms. Chung went on to have the class consider other ways people had begun with the tens. The lesson proceeded …

Discussing several examples in detail is far more helpful in developing students' mental strategies than 20 examples worked alone with no discussion.

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Key Understanding 6

There are some special calculating methods that we can use for calculations we find hard to do in our head.

Algorithms are step-by-step procedures for carrying out tasks. Most communities will expect students to be able to routinely, quickly and accurately add a column of several whole or decimal numbers, subtract one number from another, and multiply and divide at least by single-digit numbers or multiples of ten without the aid of a calculator. They may expect students to go beyond this minimum but perhaps not as quickly or automatically. Students may all have an efficient procedure but not all have the same (standard) routine. Whether or not *standard* algorithms are taught will be a decision for local school communities.

The development of any standard algorithms should *follow* the development of mental and informal written approaches and be introduced as an extension of students' existing strategies to enable them to deal with larger numbers easily. It is now understood that introducing standard written algorithms for simple numbers can be counter-productive. For example, emphasizing addition of two-digit numbers in a standard column format where no regrouping is required, actually encourages students to focus on each column separately and to lose sight of the significance of the places. This can lead to many of the errors they later make when trading is needed. Also, we should be worried if students are using a vertical algorithm to calculate 10 000 – 9998, as it suggests that they are **not** thinking and have poor number sense. It is also important that students be expected to estimate the results of calculations prior to using any procedure.

To reiterate, students should use mostly mental approaches supplemented by their own informal strategies (using diagrams or symbols) when they need the "prop" of paper and pencil. *After* students are able to confidently carry out an operation using understood mental and informal paper-and-pencil strategies, it may be appropriate to help them develop standard written methods that will be efficient for more difficult calculations.

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It should be made explicit to students that there are a range of standard methods that can assist accuracy and speed for more difficult calculations, but these do not replace other strategies. Students should choose the approach that suits the numbers, the context and their own preferences and skills.

Students should learn that there are a variety of alternative algorithms. They might, for example, compare some of their own strategies with the standard methods for ease, reliability and efficiency and reflect upon why we set certain algorithms out as we do, and the advantages and disadvantages of the setting out. They could also investigate and compare algorithms used in ancient times by the Mayans and Egyptians.

Links to the Phases

Phase	Students who are through this phase. . .
Partitioning	■ add and subtract whole numbers and amounts of money with ease and accuracy using efficient methods; these methods *may* be standard ones, but students do not have to learn standard methods in order to have reached the end of the Partitioning phase
Factoring	■ have similar facility with decimals and with multiplication and division by a single-digit number
Operating	■ use a range of efficient, although not necessarily standard, written methods for adding, subtracting, multiplying, and dividing whole numbers and common and decimal fractions

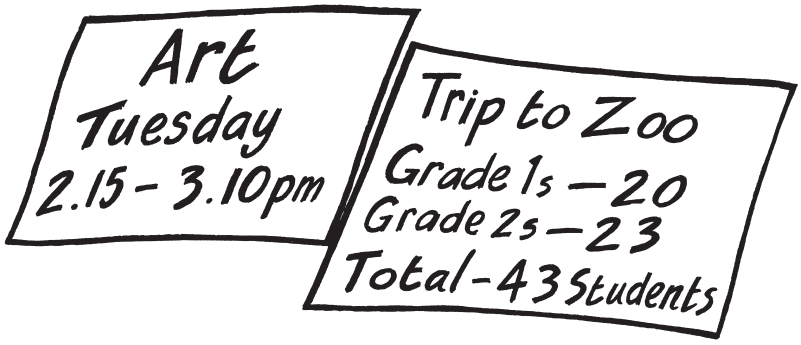
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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Making Notes

Ensure that students see you make quick notes of numbers on scrap paper to use as a memory aid when calculating class numbers, times, and so on. Encourage them to do the same, recording their numbers incidentally.



Describing

Have students describe their own methods for adding and subtracting. As each student speaks, set out the changes to the numbers as the description unfolds: *When I added 8 + 7, I took two from seven to make eight into ten.*

$$8 \overset{2}{\curvearrowright} + 7 \qquad 10 + 5 \quad \text{so} \quad 15!$$

Set out students' different strategies and draw out the use of place value to make calculating easier.

Running Totals

Have students use running totals as a routine way to get totals when adding lists of numbers devised through everyday events and games, such as card games like 21 and number cube games.

$$\begin{array}{r} 5 \\ +\ 6 \\ \hline 11 \\ +\ 4 \\ \hline 15 \\ +\ 1 \\ \hline 16 \end{array}$$

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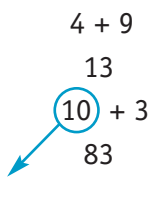
Informal Algorithms

When students are able to understand why 34 is 30 + 4, help them to see how expanding each number into standard place value partitions in their informal algorithms can make calculations easier.

34 is 30 + 4
49 is 40 + 9

40 + 30
40
50
60
70
80

4 + 9
13
10 + 3
83



Extend to three and more addends and later to three-digit numbers. Then have students use their own informal written strategies to solve subtraction problems. When they can confidently use their personal algorithms, encourage students to explore and share strategies.

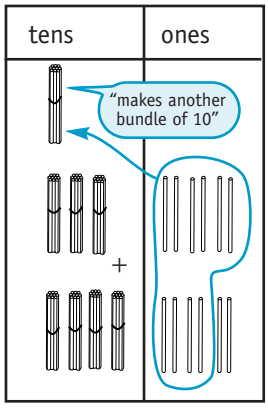
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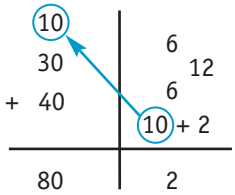
Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Trading

When students are fluent with bundling and trading games, show them how to record the numbers and operation set-out the same way as the groups of tens and ones. For example: 36 + 46





Focus students on the value of the numbers, such as 6 and 30. Extend into the hundreds column when students need to trade into the hundreds.

Sales Flyers



Using sales flyers, have students choose three prices to add and describe the different ways they have written down their numbers. Invite students to compare and debate the value of vertical and horizontal setting-out and consider whether it is easier to add the tens or the ones first.

Shopping

Similar to *Sales Flyers*, above, have students find a way of writing their numbers down to help work out how much money they have left after buying their items. Ask: How does it help to write the numbers one above the other? When dealing with large numbers, is it easier to start subtracting from the tens or ones? Why?

Try Another Way

Extend *Shopping*, above, by giving students an example of another student's thinking for them to try. For example:

56	*First I did 50 – 20 and that is 30.*
– 29	*Then I did 6 – 9 and that is – 3.*
30 – 3	*So then I did 30 – 3 and that made 27.*

Ask students to try this for several subtraction examples and say why it works.

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Other Methods

Ask students to interview their parents and grandparents (or people from other cultures) to find out how they were taught to do calculations. Ask students to present to the class a method for each operation. Ask them to compare the different methods and say why each works. Students can then select a method and use it to solve similar problems.

Everyday Problems

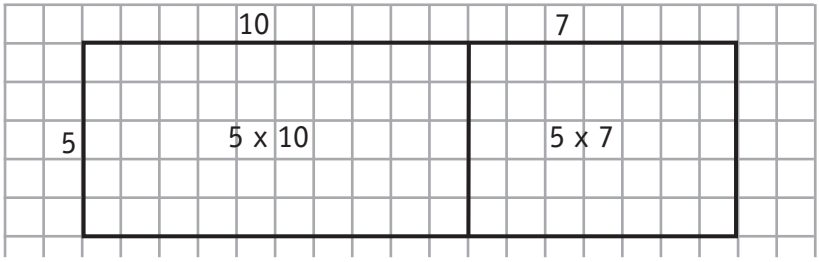
Have students ask their parents how they solve most addition and subtraction problems in their daily lives. Ask them to compare these methods with the written methods discovered in *Other Methods*, above, and say why they are different.

Making It Easier

After students solve a problem using their own informal written methods, discuss how their methods are the same or different from the methods they discovered their parents use in, *Everyday Problems*, above. Ask students when they might use their own methods and when their parents' might be more useful. (See Case Study 3, page 172.)

Multiplication Grids

Have students use 2-mm grid paper (See Appendix: Line Master 7) to represent multiplication situations. For example, Grandma bought five T-shirts for her grandchildren for their birthdays. Each cost $17. How much did they cost altogether? Help students see how five rows of 17 squares can represent five groups of $17, and that partitioning the rows into tens and sevens can give two easier multiplications.



Ask students to record the separate calculations required to solve the problem. Later encourage them to use informal written strategies from the number alone.

Class Routines

Have students develop routines as a class for commonly occurring calculations. For example, adding consecutive numbers is a calculation required to solve many problems. Ask students to work out a method that is easy and reliable. Encourage students to convince others of the merits of their routine. Make a chart of the method that has been arrived at by class consensus.

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

Comparing Strategies

Invite students to compare mental and written strategies for speed and ease in different cases, some of which are difficult to solve mentally and others which are easier. Model the steps in a range of standard algorithms to show how to find a solution. Ask: Is that how you would find the solution? Why?

Family Strategies

Have students ask their families to show them how to set out and carry out calculations, such as 554 – 376. Share these with the class and make a chart of the different ways. Ask: Did your parents do the calculation the same way as your grandparents? How are they different? Why does the "borrow and pay back" method work?

Total Scores

Ask students to work out the total scores for each of the contestants in a gymnastics competition. For example: Dale received 7.60 in round one, 8.35 in round two, 8.80 in round three; Robin received 8.15 in round one, 8.45 in round two, 9.00 in round three. Ask students to find a way of adding the scores that is easy for them. Share the methods of adding with the class. Ask: Why do you think the routine of using columns was developed? Why do you think the routine of adding from the right column was developed?

Rules Rule!

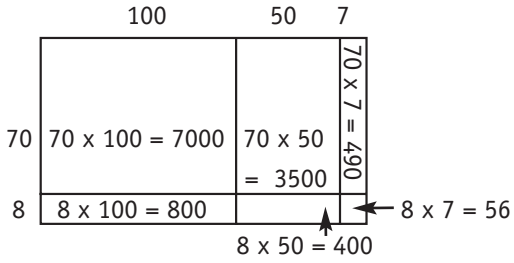
To illustrate how rules can be misapplied, present scenarios such as: Marcy said, *Right is right! You always line up the numbers on the right, get your answer then put in the decimal point*. Ask students to explain why Marcy's rule does not work for this calculation. (Hiebert and Wearne, 1994)

$$\begin{array}{r} 3.5 \\ +\ 0.62 \\ \hline 0.97 \end{array}$$

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Multiplication Grids

Have students use 2-mm grid paper (See Appendix: Line Master 7) to represent and solve multiplication calculations with larger numbers. For example: 24 x 56. Ask: Which part of the grid shows 4 x 6, 20 x 6, and so on? Later, have students use 1-mm grid paper (See Appendix: Line Master 24) and place value partitioning to multiply numbers like 78 by 157.



78 x 157 = 7000 + 3500 + 490 + 800 + 400 + 56

Grids and Base Ten Blocks

Extend the *Multiplication Grids* activity above by having students use Base Ten Blocks to represent the quantities in the different sections of the multiplication grid. Ask them to relate these quantities to the various calculations in a standard written procedure for the same multiplication. Ask, for example: where is 70 x 50 represented in the written algorithm? Try recalculating the algorithm with 157 written beneath 78 (instead of the other way round). How does this relate to the grid and the Base Ten Blocks?

Which Is Easier?

Have students compare their own solution path to that of standard methods, such as a short division algorithm. For example: To calculate 495 ÷ 5, Sharie did:

90 x 5 = 450, 45 left,
8 x 5 = 40, 5 left
1 x 5 = 5, 0 left, so A = 99

and compared this to $5\overline{)49^{4}5}$ 9 9

Ask: Which is easier for you and why? Which method would you use to compute 495 ÷ 15?

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CASE STUDY 3

Sample Learning Activity: Grades 3-5—Making It Easier, page 169

Key Understanding 6: There are some special calculating methods that we can use for calculations we find hard to do in our head.

Working Towards: The end of the Quantifying and Partitioning phases

TEACHER'S PURPOSE

Mr. Hall's school had decided that being able to add a column of four or five numbers or amounts of money easily without a calculator was a useful adult skill especially for checking the bill in a restaurant. The school felt teachers should help students develop an efficient approach to column addition, although it did not expect them all to use the same method.

Mr. Hall's grade 4 class had a lot of experience in adding two two-digit numbers. Students used a variety of mental and paper strategies, including jumping along number lines, and partitioning numbers in various ways. Many could also do some simple additions involving hundreds, such as 132 + 27. Mr. Hall thought they had the understanding needed to develop an efficient written method, but was not sure how to "motivate" it since, in most circumstances, a calculator would be readily available for the more difficult calculations.

MOTIVATION AND PURPOSE

The students were doing a science project to investigate camouflage in nature. The activity involved scattering large but equal numbers of four different colours of paper snippets on the oval. Each student then picked up as many pieces of paper as they could in a fixed time. The idea was that students would find fewer pieces of the colours that are harder to see against the grass.

Mr. Hall asked each group of four students to work out the total number of pieces they had of each colour. They counted their collections first but then had difficulty adding to find the total for each colour.

The students were able to get started, but they found adding four numbers a bit of a struggle. Almost all of them added two of the numbers, then added in the third, then the fourth and this was quite time-consuming with a lot of room for mistakes.

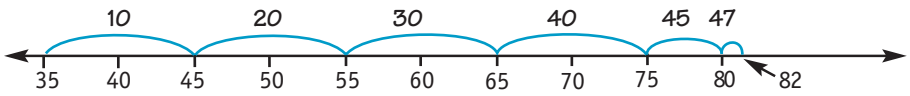
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Many groups found that the two pairs got different answers the first time and had to check to see who was right. The students found this tedious. The purpose of this particular lesson was science, but Mr. Hall had set it up so that it provided him with the opportunity to introduce efficient ways of adding "lots of numbers" or "bigger numbers".

CONNECTION AND CHALLENGE

Later, Mr. Hall reminded the students of the difficulty they had had. They agreed that usually a calculator would be available and sensible for such situations, but it is handy to have an easy way to deal with the other times. Several students commented about how their parents did "sums". Mr. Hall suggested that the easy methods for adding lots of numbers or big numbers were quite similar to methods many of the students already used for adding two numbers and it would not be much of a shift for them to learn these techniques. Mr. Hall wrote 35 + 47 on the board and asked the students to find the sum however they usually would.

After the students had a chance to perform the calculation individually, Mr. Hall asked for volunteers to explain what they had done. Craig reported using a roughly sketched number line to work out 35 + 47 and said: *I jumped along 40 to get 75 then I had seven to go so I went five more to 80 and two more to get to 82*.



Craig

35
+ 40
75
+ 5
80
+ 2
82

Without comment, Mr. Hall recorded the calculation vertically as Craig spoke (see right).

Danni said: *I added the tens and got 30 and 40 is 70, then I added five and seven which is 12, and 70 + 12 is 80 then two more is 82*.

Danni

30
+ 40
70
+ 12
80
+ 2
82

Again, without comment, Mr. Hall wrote the calculation vertically (see right).

Mr. Hall wrote up three more students' approaches, although they looked quite similar to Craig's and Danni's. He drew attention to the fact that each approach involved thinking about tens separately from ones. As they talked, Mr. Hall wrote notes to the side of each of the examples on the board.

Craig	Thinking	Danni	Thinking
35	35	35	30 + 5
+ 47	+ 40 + 7	+ 47	40 + 7
75	75	70	70
+ 5	+ 7	+ 12	+ 12
80	80	80	80
+ 2	+ 2	+ 2	+ 2
82	82	82	82

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DRAWING OUT THE MATHEMATICAL IDEA

Mr. Hall focused on Danni's approach, saying: *A lot of you have used something like Danni's methods. It is quite close to the special method for adding lots of numbers, so I want to make sure that you all understand how to do it*. Mr. Hall then wrote an addition on the board, 28 + 36, in vertical format and had a volunteer, Lynley, say what Danni's thinking would be: *Do the tens first, so 20 + 30 is 50. Do the ones next, so 8 + 6 is 14. Shift 10 from 14 to the 50 makes 60 plus four makes 64*.

Write	Think
28	20 + 8
+ 36	+ 30 + 6
50	50
+ 14	+ 14
60	60
+ 4	+ 4
64	64

Mr. Hall had written the 14 under the 50 and, when Lynley said to shift the ten from the 14 to the 50, he remarked casually that it was like going back and doing the tens again. Mr. Hall then asked students to make up one for themselves, set it out vertically, do it, check their work with a partner, then practise several more. As Mr. Hall moved around, he reassured himself that all but one or two students (to whom he returned later) were confident—for many it was close to what they had been doing, even if they had not set it out so systematically.

16
18
37
+ 22
70
+ 23
90
+ 3
93

Mr. Hall decided to move on to adding several numbers and wrote 16 + 18 + 37 + 22 on the board, choosing an example with a sum under 100. The students all wrote the numbers down in a column, and they talked though the process: *Do the tens first, then do the ones, then shift any extra tens across* ... They proceeded happily with this, and did not appear to notice that they were shifting 20 across to the tens rather than the ten they had dealt with so far. Mr. Hall gave them another to try. Several students asked if they could skip the last bit and go from 70 + 23 directly to 93. Mr. Hall asked the opinion of the class, and all agreed it seemed sensible.

Mr. Hall had decided that they had done enough for one day, when Chris said that his parents said you were supposed to do the ones first! Mr. Hall then asked the students to talk in their group about why they thought Chris's parents had learned to do the ones first and which way was better.

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Somewhat to Mr. Hall's surprise, several groups suggested that when you were writing it down you could save a line if you did the ones first because you could add in the 20 right away! Mr. Hall asked other students their view, and several agreed that if you were writing it down, doing the ones first saved a line. He suggested that they all try doing the ones first, but that those who preferred to set it out in full should, and those who preferred to shorten it should.

In full	Shortened
16	16
18	18
37	37
+ 22	+ 22
23	23
+ 70	93
93	

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Key Understanding 7

We can calculate with fractions. Sometimes renaming fractions is helpful for this.

Adding and subtracting fractions should follow naturally from understanding what fractions mean, visualizing fractional parts and counting backwards and forwards in fractional amounts. Students should represent fractional amounts in materials and pictures and, for example, count forward: one-fifth, two-fifths, three-fifths, four-fifths, one, one and one-fifth, and so on. This leads quite naturally to the idea that "two-fifths add one-fifth is three-fifths". Students initially work in words, both oral and written, and use abbreviations, such as 2 fifths + 1 fifth = 3 fifths, to record the results of their thinking. When they are confident with the ideas, the conventional recording, such as $\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$, can be used.

Adding three-fifths and four-fifths could be seen in the same way, by imagining or drawing a diagram or "counting forward" to get seven-fifths or one and two-fifths. Alternatively, students might imagine partitioning the four-fifths into two groups of two-fifths, combine one group with the three-fifths to make the whole, and hence get one and two-fifths.

Learning a procedure such as "add the numerators and leave the denominators unchanged" is initially unhelpful since it focuses upon the two separate parts of the fraction and encourages students to think of fractions, such as $\frac{2}{5}$, as two numbers rather than one. This often results in students producing answers, such as $\frac{1}{2} + \frac{1}{2} = \frac{2}{4} = \frac{1}{2}$, which suggest that they have not thought of the fractions each as a number in itself, but rather have focused on the numerator and denominator separately.

For fractions with unlike denominators, students need to build upon their understanding that the same fractional amount can be written in different ways, so that, for example, one-half is the same amount as two-fourths. Given extensive experience with partitioning and equivalence, adding and subtracting unlike fractions follows naturally, so long as the relationship between the denominators is easy to see. That is, the partitioning and equivalence knowledge required to work out sums and differences, such as $\frac{1}{2} + \frac{1}{4}$ and $2\frac{3}{4} - \frac{1}{2}$ is reasonably intuitive because the $\frac{1}{4}$ is contained in the half and students generally can readily visualize $\frac{1}{2}$ as two-fourths. When students first begin to add and subtract fractions, the emphasis should be on visualizing each fraction as an entity and

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the fractions involved should be those which are relatively easy to imagine or draw in simple diagrams. To add $\frac{1}{3}$ and $\frac{1}{4}$ is much more difficult, since a common unit that is not instantly recognizable needs to be found. Students will need considerable experience in choosing how many parts they need to partition wholes into so that both fractions can be represented. It is not obvious to students and cannot be rushed.

Finding a fraction of a number, such as $\frac{1}{3}$ of 27 or $\frac{1}{3}$ of 28 or $\frac{1}{3}$ of 28.2, involves partitioning the number into equal parts and taking the appropriate number of those parts (in these cases, taking one part of three). Often factors are useful for this. Again the emphasis should be on the partitioning of collections and wholes into easily visualized or drawn parts.

Students will need developmental activities for several years in order to develop computational flexibility with fractions.

Links to the Phases

Phase	Students who are through this phase. . .
Partitioning	■ can count forward and back orally in unit fractions (one-sixth, two-sixths, three-sixths) and write these in words ■ are able to add and subtract simple fractions presented in words, although they may not be able to add or subtract the same fractions presented in conventional symbolic forms
Factoring	■ will be able to carry out such computations, even when presented symbolically, but they will use most mental methods based on counting forwards and backwards in fractions
Operating	■ can mentally add and subtract fractions that are easily visualized or drawn or involve well-known equivalencies ■ can record the stages in adding and subtracting fractions that they cannot complete mentally ■ can find unit fractions of whole numbers mentally and find proper fractions of quantities either mentally or with written support

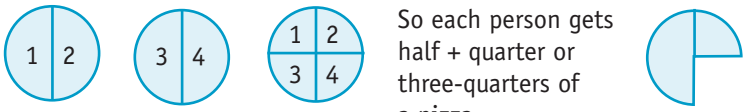
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Sample Learning Activities

Grades 3-5: ★ Introduction, Consolidation or Extension

Sharing Pizzas

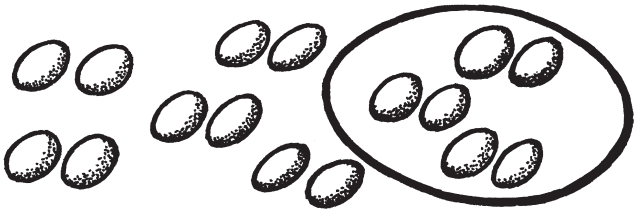
Have small groups of students share equally a number of paper circles that represent pizzas to say what fraction of a circle each will get. For example: Four students share three pizzas, using diagrams to plan how they could divide the paper pizzas. Ask students to cut the circles into parts and combine them to make their share, then record the answer in shorthand.



So each person gets half + quarter or three-quarters of a pizza.

Partitioning Collections

Ask students to partition and combine collections of materials and solve simple number problems. For example: Partition 16 beads into eighths then show three-eighths. Ask: How many eighths will be left if you give away three-eighths? Ask students to say the answer: *If you take away three-eighths of the beads, five-eighths are left.* Help students record in symbols, drawing out that the collection of beads is the whole or "one" ($1 - \frac{3}{8} = \frac{5}{8}$). Ask: If I gave away four-eighths and then gave one-eighth to my brother, how many eighths have I given away? Write in shorthand (4 eighths + 1 eighth = 5 eighths). Record in symbols.



Practical Fractions

Invite students to use materials to solve simple practical problems involving fractions. For example: If all 32 of us use half a piece of paper each, how many whole pieces of paper will that be? Ask students to reflect on what they did and record it as a number sentence. Ask: How can you write a sentence to show you needed a half, 32 times?

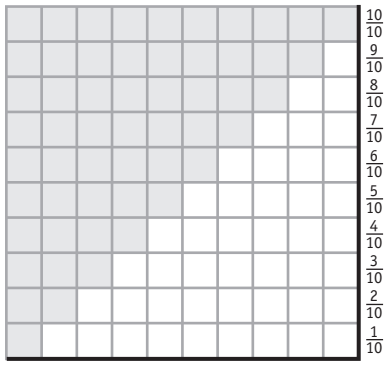
Partitioning Circles

Have students partition circles into halves and/or quarters to work out examples, such as $1\frac{1}{2} + 1\frac{1}{2}$ or $2 - \frac{3}{4}$. Encourage students to compare their approaches.

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Arrays

Using a 10 x 10 array (See Appendix: Line Master 17), ask students to colour in ten squares on the first line, nine on the next, eight on the next and so on. Use this to find the difference between fractions, such as $\frac{10}{10}$ and $\frac{4}{10}$, $\frac{7}{10}$ and $\frac{2}{10}$. Have students write a number sentence to record their findings, such as $\frac{7}{10} - \frac{2}{10} = \frac{5}{10}$. Ask them to write their own rule for subtracting fractions with like denominators. Extend to other denominators.



$\frac{7}{10} - \frac{5}{10} = \frac{2}{10}$

Number Line

Have students use a number line to combine fractions. For example: Show $1\frac{1}{2} + 1\frac{1}{2}$ by first finding $1\frac{1}{2}$ and then counting on to find the position of $1\frac{1}{2}$ more.

Making a Whole

Ask students to find the different fractions that can be put together to make a whole during area activities. For example: When finding the area of a leaf, say: That is about $\frac{3}{4}$ of a square and that one is about $\frac{1}{4}$, so we could count the two together as one.

Double the Quantity

Have students decide how to double quantities in cooking, for example, $\frac{3}{4}$ of a cup of sugar. Ask them to talk about the ways of finding out the answer. For example: $\frac{3}{4}$ could be seen as $\frac{1}{2}$ and $\frac{1}{4}$ so when doubling, the $\frac{1}{2}$ and $\frac{1}{2}$ could be put together and then the $\frac{1}{4}$ and $\frac{1}{4}$ to make $1\frac{1}{2}$.

Pattern Blocks

Ask students to use Pattern Blocks to find fraction combinations that add to one. For example: If the hexagon is used as the whole, then one blue rhombus and four triangles equals one. Have students write this, $\frac{1}{3} + \frac{4}{6} = 1$, and say how it is possible to add thirds and sixths by comparing the size of the rhombus to the size of the triangles, or vice versa.

Equivalent Fractions

Ask students to investigate equivalent fractions with Pattern Blocks. For example: If we make a shape with a blue rhombus and a triangle, how much of a hexagon do we have? What fraction of the hexagon does the triangle show? The rhombus? Have students write an addition sentence to show their equivalent fractions, such as $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ or $\frac{1}{6} + \frac{2}{6} = \frac{1}{2}$. They then say how much of the hexagon is not covered and write a number sentence, such as $1 - \frac{1}{2} = \frac{1}{2}$.

Passing Time

Have students use a clock with movable hands to find out how much time has passed. For example: Spelling took a quarter of an hour and math took a quarter of an hour. How much time did we use? Ask: How many minutes in a quarter of an hour? Does it help to add minutes instead of quarter hours?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Check for Sense

Have students focus on the meaning of fractional quantities to check that their calculations make sense. For example: Connie wrote $\frac{3}{4} + \frac{1}{2} = \frac{4}{6}$, saying this was how long she had watched TV. One show was $\frac{3}{4}$ hour and the other $\frac{1}{2}$ hour. Does Connie's result make sense?

Dividing Whole Numbers by Fractions

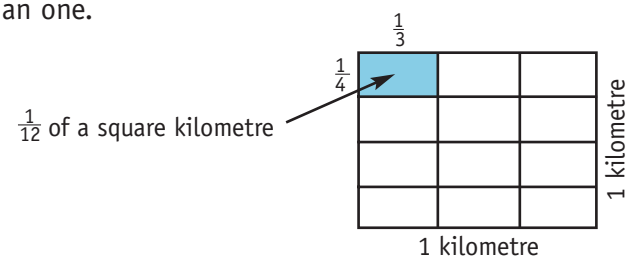
Ask students to divide whole numbers by fractional quantities using linear representations, such as string and number lines, in situations such as: We need $\frac{3}{4}$ m lengths of ribbon. How many can we cut from 5 m?

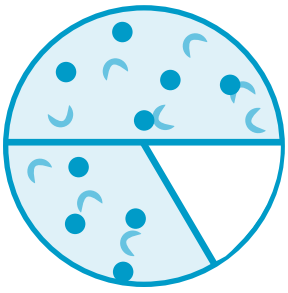
Leftovers

Have students relate division to the fraction notation where the quantities to be shared are not whole amounts. For example: three-quarters of a pie was left in the fridge. Three students were to share the piece. What fraction of the leftover pie will they each get? ($\frac{1}{3}$) What fraction of the *whole* pie will they get? ($\frac{1}{4}$) Draw out that $\frac{3}{4} \div 3$ is the same as one-third of $\frac{3}{4}$, which is $\frac{1}{4}$. Have students solve similar problems where a fractional part needs to be shared.

Area Problems

Use an array representation to solve fraction area problems. For example: What is the area of a rectangle that measures $\frac{1}{3}$ km by $\frac{1}{4}$ km? Draw a diagram to show the calculation. Ask students to explain why the result is less than one.





Pizzas and Pies

Have students use their own methods to solve problems involving fractions and "share and compare" strategies.

- *The pizza problem*. Show students a diagram of two pieces of pizza on a plate: a half and a third. Say: The whole pizza was the same size as the plate. Point to each piece and ask: What fraction of the whole pizza is this? What fraction of the whole pizza are these two pieces together?

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- *Three-quarters of a pie.* Three-quarters of a pie was left in the fridge. Six students have to share the piece of pie. Ask: What fraction of the whole pie is each share? ($\frac{1}{8}$) What fraction of the three-quarters piece of pie is each share? ($\frac{1}{6}$)

Pattern Blocks 1

Have students add fractions with Pattern Blocks, using two yellow hexagons taped together as the whole. They check which pieces can be used to show fractional parts (green triangle = $\frac{1}{12}$, blue rhombus = $\frac{1}{6}$), list ways in which two different coloured pieces can make one whole, and write as a sum, such as $\frac{2}{6} + \frac{8}{12} = 1$. A partner checks by covering the whole with the appropriate pieces.

Patterns Blocks 2

Extend *Pattern Blocks*, above, to add and subtract simple fractions, such as $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{4} + \frac{2}{6}$, $\frac{1}{4} + \frac{5}{6} + \frac{1}{12}$, $\frac{1}{2} - \frac{2}{6}$, $\frac{1}{4} - \frac{1}{12}$. Ask: How did renaming or trading the pieces help you find the solutions?

Finding Fractions

Ask students to find fractions of whole numbers. For example: Find $\frac{3}{4}$ of 24 straws and use materials or diagrams to explain the process you used.

Multiple Slices

Have students decide how many partitions are needed to be able to easily take two different fractions of the same thing. For example: How many slices could I cut the cake into so it is easy to take a third? (3, 6, 9, 12, ...) To take a quarter? (4, 8, 12, 16, ...) Ask: How many so it is easy to take a third and a quarter? Draw out that the number of pieces has to be a multiple of each number. Build on this and extend to find the fraction of the slice that is gone if we eat $\frac{1}{3} + \frac{1}{4}$. (See Multiple Slices activity, page 140.)

Equivalent Forms

Have students use diagrams, number lines, counters or arrays to show that fraction number sentences can be written in equivalent forms without changing the quantity. For example: Jim said, "I have to calculate what $\frac{1}{3} + \frac{1}{4}$ is, so I'm going to change it to $\frac{3}{12} + \frac{4}{12}$ to make it easier to work out." Ask students to show why he can do this. Rename fractions in other calculations, such as $\frac{3}{4} - \frac{1}{3}$, to make them easier to work out.

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Key Understanding 8

Rounding, imagining a number line, and using properties of numbers and operations help us to estimate calculations.

Good estimation and approximation skills enhance our ability to deal with everyday quantitative situations. Students should be provided with ample opportunity to decide whether an estimate of the result of a calculation is sufficient and, if so, how close the estimate needs to be and of what form. They should also learn to use estimation routinely to judge whether the results of more accurate calculations are reasonable and to determine the order of magnitude of a result.

Approximate calculations are accurate calculations based on simplified numbers. They often involve rounding and single-digit mental arithmetic and powers of ten. Thus, to approximate 27 x 16, we could round to the nearest ten in either direction and say that the result is between 20 x 10 = 200 and 30 x 20 = 600. Alternatively, we might get a bit closer by saying it is a bit more than 25 x 16 = 25 x 4 x 4 = 400. To check decimal places for 25.14 x 3.5, we might think "it'll be between three 25s and four 25s, which is between 75 and 100". To estimate gate-receipts for each day, we might take the number of people who came through the turnstile today (343) and round it to 340, saying the gate-receipts should be about 340 times the cost of admission, $8.

Whether we round to the nearest one, five, ten, or 100, or half depends upon the context. For estimating 0.61 x 558, a sensible strategy might be to find half of 600, yet some students will round the 0.61 to one and 558 to 600 and draw the inappropriate conclusion that the approximate answer is 600. Often, rounding is not the best strategy for approximating answers. For example: For 79 ÷ 9 we might choose "nice numbers" and say it is a bit less than 81 ÷ 9. We might also compensate. For example: 43 x 51 is about 40 x 50 = 2000 and three more 50s, so 2150 is closer.

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Whether we round "up", "down" or "to the middle" also depends upon the context. To ensure you have enough money for a series of purchases, you will probably round up rather than to the middle. Repetitive exercises in rounding numbers out of context are likely to lead students to think that there is a single right way to round. They may also see rounding as just another meaningless mathematical exercise, rather than a practical tool for everyday estimation.

Experiences should also be provided to enable students to judge about how much an answer will be without calculating. For example, after sketching a number line (say from zero to ten) and marking 2.6 on it approximately, students could judge three groups of the distance from zero to 2.6, or one-third of the distance. This will give an estimate of 3 x 2.6 and $\frac{1}{3}$ x 2.6. With experience it is possible to estimate in the "mind's eye" with sufficient accuracy for many purposes.

Students should also learn to use their knowledge of properties of numbers and operations to judge the size of an answer. For example, they should be able to say that 0.2 x 0.3 must be smaller than 0.2 (since it is multiplied by a number smaller than one) thus it cannot be 0.6 (a common error).

Links to the Phases

Phase	Students who are through this phase. . .
Quantifying	■ estimate in simple ways to check two-digit additions and subtractions, saying for example, that 16 + 19 cannot be 25 because it has to be more than 30
Partitioning	■ estimate both sums and products by rounding to single-digit numbers or simple multiples of ten and visualizing on a number line, although they may need prompting and support
Factoring	■ estimate both sums and products by rounding to single-digit numbers or simple multiples of ten and visualizing on a number line without much help *For example:* A student may say that 47 + 49 will be more than twice 45 (90) and less than twice 50 (100). They will also say that 16 × 9 is more than 90 (10 × 9) and less than 160 (16 × 10).

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Sample Learning Activities

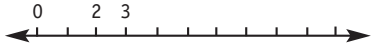
K-Grade 3: ★ Introduction, Consolidation or Extension

Estimating Collections

Invite students to look at a collection of things and estimate if there will be more than, less than or enough for everyone in a small group to have one each. For example: Spread seven glue sticks in front of a group of three students. Ask students to say whether there will be too many or not enough.

Benchmarks

Have students use small numbers, such as two, as benchmarks on a number line and say where other numbers would be.



Pencils

Ask students to partition and round up amounts to tens to estimate how many packs will be enough when things come in packs of ten. For example: Everyone needs a pencil each and the pencils come in packs of ten. Ask students to suggest ways to work this out mentally. Draw out partitioning into decades and ones, such as 32 = 30 + 2. Three tens and two ones. Then focus on rounding up. Ask: How many packs must we buy? Will three be enough? Why? Why not? Ask students to justify their estimates, using diagrams to support their idea.

Bulbs

Have students use partitioning into fives or tens and rounding down to decide how many, when they need to make equal groups. For example: Mark's granddad gave him 28 tulip bulbs and five flower pots. He thinks that will be about six bulbs in each pot. Ask: What do you think? How could you check? Are there any more ways we haven't thought of?

Number Scrolls

When skip counting with a calculator, ask students to record the numbers on cash register tape and say what number will come next and what number they might reach at the end of the strip. Ask: How did you decide?

Money

Have students estimate using doubling and rounding. For example: Collect catalogues and rewrite the prices for items in multiples of ten cents or dollars, such as 40 cents, 50 cents, 90 cents, $1.10. Ask students to select an item and say whether $2 will buy two of them. Focus on doubling and rounding. Extend by having the students choose several items and then either $1, $2 or $5 to cover the cost.

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Number Lines

Ask students to imagine numbers on a number line or 100-chart to decide whether to round up or down to the nearest ten and then to use this to help estimate answers to problems. Ask: When should you round up (down)?

Calculator Test

Have students use the calculator to test which way of rounding will give the best estimate. For example: For 268 + 425, try 200 + 400, 300 + 400, 300 + 500 and 200 + 500.

Shopping Estimates

Given a shopping context and real prices, have students round to decide if they have enough money for specified items. Ask them to share their estimations and tell whether they overestimated or underestimated. Discuss the situations where we would want to overestimate and underestimate and our strategies for doing this.

Multiplying by 26

Have students use a calculator to find a number that can be multiplied by 26 to give an answer between 100 and 150. Ask: Which number did you start with? Why? Did you then choose a bigger or smaller number? Why?

Guess the Answer

Ask students to say two numbers that an answer to a calculation should be between. For example: Five packages of paper clips at $2.20 should cost between $10 and $15. Ask: Can we say that it will definitely be more than $11, or less than $14? Try to get the range as small as possible to make a close estimate.

Supermarket Receipts

Have students estimate to match supermarket receipts with the totals that have been cut off. As a class, consider which strategies were commonly used to estimate and why. Repeat the activity using a different strategy from the one first used. Ask: Do supermarkets round up or down? Why?

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Grades 3-5: ★ ★ Important Focus

Working with Wood
Have students decide how close their estimate needs to be when working out how much wood they will need to make a picture frame for their artwork. Ask: Should the answer be within 10 m, 1 m, 50 cm, 10 cm or 1 cm? Why?

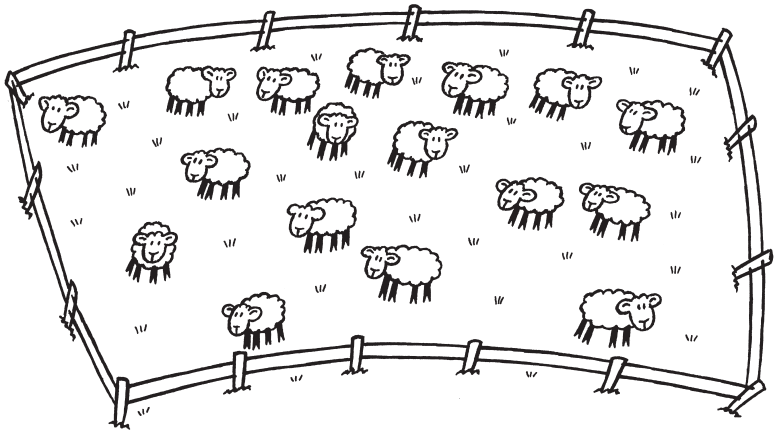
Place Value
Ask students to say which answer must be right using estimation based on place value. For example: 5 x 15 = 7.5, 75 or 750; 24 x 61 = 1464, 126, 1224 or 142; 39 x 0.5 = 195, 155, 19.5 or 1595.

When to Estimate?
Have students brainstorm to create a list of situations where they have used estimation instead of a precise calculation. Ask them to interview parents to add to the list. Ask: How close did the estimate have to be to the exact answer? Why?

Does It Make Sense?
Without actually doing the calculation, involve students in thinking, talking and writing about whether an answer makes sense. Explain why. For example: 304 x 2 = 600; 3 x 345 = 1050; 66 ÷ 7 = 10 r 4; 238 x 5 is greater than 1000.

Estimating Collections
Invite students to estimate how many there are in a collection, using either pictures of large amounts of things, or visual images. For example:
- How many parents came to the special assembly?
- How many sheep in a paddock?
- How many trees around the school playground?
- How many people at a hockey game?

Compare estimations and reasons for the amounts decided. For example: At assembly, I know that about 20 people can stand across the back and there were people behind them, so that makes about 40.



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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Exact or Approximate?

Have students decide whether exact or approximate calculations are appropriate. For example:

- You know that a can of paint covers 16 square metres and you want to paint your bedroom.
- The pears cost $1.45 kg. Jane has $2\frac{1}{2}$ kg on the scales. Would the $4 she has be enough to pay for it?
- Check that $87 collected from 58 students paying $1.50 each for swimming lessons is correct.

Ask: What calculation will you use for each? Will you round, or use an upper limit or a lower limit, to approximate the result? Why?

Bigger or Smaller

Have students judge the effect of operations on numbers to say whether the answer will be bigger or smaller than the first number, and explain why in each case. For example: 26 x 2, 23 x 0.95, 24 ÷ 2, 24 ÷ 48, 24 ÷ 0.5, 24 + 2, 24 + 0.45, 24 – 2, 24 – 0.2. Ask: Do any answers seem surprising? Why?

Estimating Fractions

Have students choose a unit (say, a decimetre) and use it to make fraction strips of six units (say, six decimetres), marked to show quarters in one colour and thirds in another colour. They then cut a number of paper strips of different lengths (but shorter than their fraction tape measure) and fold to find one-third of each strip. They then place their paper strips alongside their fraction tape measure and say how long their strip is and how much a third of the strip is. For example: The first strip is $2\frac{1}{4}$ long and one-third of that is about $\frac{3}{4}$; the second strip is $3\frac{2}{3}$ long and one-third of that is between $2\frac{1}{4}$ and $2\frac{1}{3}$.

Estimating Thirds

Ask students to estimate one-third of different amounts by judging directly on the fraction tapes made in the previous activity. For example: Judge "by eye" one-third of $4\frac{1}{4}$. Check by folding. Continue until they can confidently estimate one-third, then estimate two-thirds (one-third from the opposite end). Repeat for other unit fractions. Draw from a class discussion that estimating the size of the fraction can help us estimate and check multiplication and division.

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Grades 5-8: ★ ★ Important Focus

Fraction Tapes

Use cash register tape to design number lines of the same length, which extend from zero to about ten. One number line can show the halves, such as 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2 ... , another the thirds, and others quarters and fifths. Have students look at their "half" number lines and estimate where $\frac{1}{2} \times 4\frac{1}{2}$ would be. Ask: How did you think about that? Repeat with their thirds number line and visualize where $\frac{3}{4} \times 2\frac{1}{3}$, $\frac{1}{2} \times 4\frac{2}{3}$ and the like would be. Repeat with similar questions using the other fraction tapes.

Adding Fractions

Using the number lines made in *Fraction Tapes*, above, students can cut a piece of cash register tape to match the "length" of each fraction. They then stick the two together and use the fraction tapes to find their best estimate of the total. Do this for several and then challenge students to do the same thing in their "mind's eye", estimating the lengths.

Other Strategies

Ask students to use their fraction tapes to find other strategies for estimating the addition of two fractions. For example: For $3\frac{1}{2} + 2\frac{3}{4}$, begin at $3\frac{1}{2}$, jump along two and then another quarter.

Is It Reasonable?

Have students decide whether situations are reasonable. For example:

- When eight people won a million dollars in a lottery, they got $12 500 each.
- Enough bottles of water for the class would cost about $30.

Best Estimates

Ask students to write calculations that will give the best estimate. For example: 9572 + 6956; $\frac{9}{31} + \frac{4}{9}$; 537.05 – 38.45; 297 x 378; 2.123 x 4.89; 165 ÷ 9. Ask students to describe how they simplified the numbers.

Upper and Lower Limits

Have students identify the upper and lower limits for a calculation. For example: Start with visual situations, such as estimating the number of 2-mm squares on a page, the number of books on a shelf, or the number of 100s and 1000s on a piece of bread. Ask students to show how they worked out what the lower limit and the upper limit is for each. Extend to actual calculations, such as 125 x 42.

Posting Calculations

Extend *Upper and Lower Limits*, above, to "Posting Calculations." Label three or four boxes with a range of limits for the solutions to calculations. For example: 1–99, 100–199, 200–299. Have students write number sentences on cards for others to "post" into an appropriate box. Extend to include number sentences using decimals and common fractions. Change the limits to match.

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CASE STUDY 4

Sample Learning Activity: Grades 5-8—Fraction Tapes, page 188

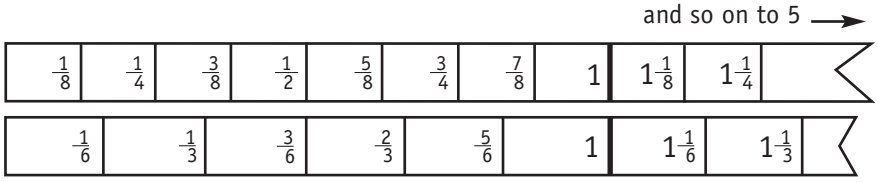
Key Understanding 8: Rounding, imagining a number line, and using properties of numbers and operations help us to estimate calculations.

Working Towards: The end of the Factoring and Operating phases

TEACHER'S PURPOSE

During an activity where Ms. Dean's class of 11- and 12-year-olds needed to scale down quantities of ingredients in a recipe, she noticed many had difficulty finding a fraction of the quantities. While they could easily calculate one-third of three tablespoons of margarine, they could not find one-third of quantities like two teaspoons of salt or $2\frac{1}{2}$ kg of cheese. When they tried to use numbers to make the calculations, they came up with quite improbable results without noticing something must be wrong. Ms. Dean decided to help them estimate the results of such calculations using a "tape measure" representation for fractions.

The students had previously built up fraction number lines using cash register tape. Each was five units long, the unit being a decimetre. One tape was labelled halves, quarters and eighths, and another thirds and sixths.



Ms. Dean decided to use these as the basis for some work on estimating fractions to try to build students' number sense about the results of calculations.

CONNECTION AND CHALLENGE

Ms. Dean began by asking each student to cut a length of cash register tape less than half a metre long. She then asked them to use their fraction number lines to find the length of their paper strip in decimetres and to record this on the end of the strip. Since few paper strips exactly matched one of the fraction marks on the number line, this required estimation. Students compared their strips to both number lines to decide whether "eighths" or "sixths" worked better and then worked in small groups to check each other's estimates. As Ms. Dean moved around the room, she saw the students understood the task and were able to use the number lines to estimate the length of their strips in decimetres.

Students commonly visualize fractions as "slices" of shapes like cakes or pies, which makes the result of calculations like $\frac{1}{3}$ of $2\frac{1}{2}$ very difficult to think about. How do you visualize $2\frac{1}{2}$ pies in three equal parts? On the other hand, a "tape measure" representation enables students to see $2\frac{1}{2}$ as a single continuous quantity, a whole that can easily be visualized in three equal parts. At the same time, the "tape measure" mental image for fractions more closely matches the abstract idea of a number line, giving fractions meaning within the whole number system.

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Ms. Dean then asked students to look at one strip and, without folding, mark where they thought one-third of it was. Each strip was then passed around the group so that every group member could mark their estimate of a third on it. Once the strip had been around the group, it was folded to check where one-third actually was. This was compared to students' estimates, talking about how they mentally made their estimates just by looking at the strip. They then repeated the process for the next group member's strip, trying to improve their "eye" for one-third.



Once each group had their estimates and the actual third marked on their strips, Ms. Dean pinned two of the fraction number lines on the board and asked for someone to bring their strip to the front. Terri volunteered and Ms. Dean asked her to tell everyone the length of her strip and show where a third of it was. Ms. Dean then asked her to use the number lines she had pinned on the board to find the length of the third of her strip in decimetres. Since the third did not exactly match a fraction mark on the number line, this required another estimate.

The total length of Terri's strip was a little less than $2\frac{3}{4}$ decimetres, and she now estimated her one-third to be between $\frac{5}{6}$ and one, but closer to $\frac{5}{6}$. *So what have we discovered?* Ms. Dean asked. After a bit of a struggle Terri said doubtfully, *A third of the strip*. Ms. Dean asked: *And how long was the strip?* Still somewhat hesitantly Terri said: *Nearly* $2\frac{3}{4}$. Ms. Dean wrote with a flourish on the board: So $\frac{1}{3}$ of nearly $2\frac{3}{4}$ = a bit more than $\frac{5}{6}$.

Ms. Dean then called on another volunteer and repeated this with a new length. The rest of the students then used the process on their own strips, writing down their results as number sentences. Ms. Dean then wrote a range of fractions on the board, and challenged them to use whatever strategy they wished to estimate a third of the various numbers. Some cut a length of tape to match the number and folded it carefully to make thirds, while others judged directly from the fraction number lines—some folded the actual number lines and other "just looked". Ms. Dean encouraged them to try to judge first, even if they then folded, since the whole point of the activity was to improve their confidence in their capacity to judge a given fraction of any length.

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After the class found a third of several different numbers, Ms. Dean asked the students to each write down a fraction between one and five and ask their partner to mark the number on a number line and estimate a third of it, without cutting and folding strips. This they did with ease. The class then talked over the strategy they used and Allen suggested: *It's like you had to forget about the* $2\frac{1}{2}$ *and think of it as a whole strip that you make into thirds like it's a licorice shared in three pieces, then you can think about where that comes to on the fraction strip*.

OPPORTUNITY TO PRACTISE

Students continued to practise the process, challenging themselves to see how well they could visualize and estimate a third of different numbers between zero and five just by looking at their fraction number lines. They then folded and measured to see how reasonable their estimates were.

Ms. Dean extended the opportunity for estimating with fractions by suggesting they do the same thing for finding one-quarter. She hinted that they might want to begin by practising finding one-quarter "by eye" on blank strips of tape. The students used language like *in between..., a bit more than..., close to...* to identify their results. Some even began to use their knowledge of fractions to estimate twelfths and sixteenths by halving the spaces. For example, Kate said: *It looks about halfway between* $\frac{5}{6}$ *and one, so it must be about* $\frac{11}{12}$*, because twelfths are halfway between sixths*. Over the next several days and weeks, Ms. Dean often asked students to estimate a fraction of another fraction, to provide practice using the strategies they had developed.

Future Directions: ***Students made good use of their fraction strips in future lessons involving operations and calculations with fractions. They used them as practical tools to add and subtract fractions with unlike denominators, compare multiplication situations like*** $5 \times \frac{1}{4}$ ***to*** $\frac{1}{4}$ ***of five, and divide by fractions, such as how many*** $\frac{3}{4}$ ***in four? All of which continued to support the development of their ability to estimate and make sense of calculations involving fractional numbers.***

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Key Understanding 9

To use a calculator well, we need to enter and interpret the information correctly and know about its functions.

Students should have ready access to calculators from the earliest years of schooling. Calculators help students to develop many mathematical concepts earlier and more thoroughly than they can if they do not have them, to work on more realistic applications, and to develop calculator skills that they will use in everyday life. For laborious or repetitive computations, calculators are the sensible choice. The regular use of calculators is assumed throughout the curriculum. This Key Understanding, however, is about the conceptual and technical knowledge needed to use them to calculate efficiently and correctly.

At the simplest level, students need to learn through exploration that digits are entered from left to right as you read them, thus 24 is entered 2 then 4. Entering 3 + 7 is intuitive on most calculators if the equal sign is remembered: 3 + 7 =. Entering 16 + 7 x 3 may be more difficult, depending upon the type of calculator used. According to the order of operations, multiplications and divisions are completed before additions and subtractions, so 16 + 7 x 3 means 16 + (7 x 3). Simple four-function calculators usually carry out calculations in the order in which they are entered, so if 16 + 7 x 3 was entered as in the order shown, the 16 + 7 would be calculated first, giving the wrong answer. Students should learn how their calculator works and enter calculations in the appropriate order. Older students should realize that calculators differ and develop the habit of checking how the calculator they are using operates.

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Students may experience entry difficulties even with simple computations. Some will enter 3 ÷ 15 = for $3\overline{)15}$. Others will always enter the larger number first in a subtraction or division, believing that you cannot "take a bigger from a smaller" or "divide a smaller by a bigger". Some who know that there are two 50 cents in a dollar and hence sixteen 50 cents in $8, will enter the calculation as 8 ÷ 50 or even 50 ÷ 8. Thus, using a calculator often helps to expose conceptual misunderstandings. The cognitive conflict caused when the calculator answer does not fit their common sense can provide a helpful stimulus for students to overcome misconceptions and inconsistencies in their thinking.

Students should learn to use the various features of their calculators effectively: repeated operations (using the constant function on many calculators—to count by threes you press 3 + = = = ...), memory and bracket facilities, change of sign, and so on. Interpreting the results of computations, such as 1.5 as $1.50, a negative answer when finding a difference, remainders from a division, can be conceptually demanding and requires special attention. As indicated in Key Understanding 8 on page 182, students should also learn to use estimation for effective calculator use, providing a way of checking that a problem has been correctly formulated for the calculator, and the keystrokes have been executed correctly.

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Sample Learning Activities

K-Grade 3: ★ ★ Important Focus

Exploring Calculators

Have students use calculators to explore, press keys and discover what happens. Focus students on watching the display to notice what happens each time they press a number key. Have them take turns to read out the digits on the display for others to make the same numbers.

Clearing

Encourage students to become familiar with the function of particular keys, such as C/CE. After a student discovers how to clear numbers, ask them to show all students the steps. To practise, they key in 1, press the clear key to clear it back to zero. Continue in order through the numbers to 12. Later as they enter calculations, such as 24 + 36, show them how pressing this key once clears the 36 and pressing it twice clears all back to zero. Practise this through games, such as "Simon Says".

Explaining to Others

When students find something that always happens on the calculator, have them explain it step-by-step to others. For example: When you have a sequence of numbers and press C/CE, it changes the numbers to zero.

Correct Terms

Ask students to use common correct terms to describe what they do on the calculator, and see what happens, such as press, key in, display, clear. At first students will use their natural language to describe the function keys that are beyond their everyday use. For example: They might say "the squiggly thing" for the square root sign. Over time, support them to use accurate and conventional terminology.

Recording Device

Have students use the calculator as an informal recording device to enter multi-digit numbers such as telephone numbers, dates, game scores, to read and for others to copy. Ask: Which number (digit) did you put in first? Why?

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Counting Device

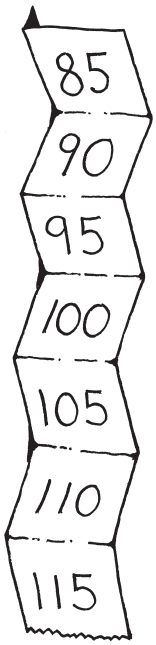
Ask students to use the calculator as a counting device. For example, use the constant function as shown in Case Study 5 on page 200 to:

- Count collections of things by ones or in groups. For example: Count ants on an ant trail by ones or another class entering a room by twos.
- Discover and explore negative numbers by counting backwards from a given number.

Constant Function

Have students watch numbers increasing into the hundreds, using the constant function to skip count.

- Number scrolls: Invite students to record the numbers vertically on long strips of paper. Interrupt the count at times for students to predict the next number.
- Develop "number lines", such as lily pads, for a frog to jump along.



Computational Tool

Use the calculator as a computational tool. Have students choose which operation key to use when numbers are too large to solve mentally. Students may begin to use the ÷ key for sharing problems, such as eight teddy bears need to share 74 cakes. How many cakes will each bear get? Share 74 cakes among eight bears.

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Prior Knowledge

Have small groups of students write down what they know about the calculator and how it works. Invite the groups to share this information with the whole class, and make a chart of the results. Add to this throughout the year as the students extend their knowledge.

Count Forwards and Backwards

Invite students to explore the use of the constant function in counting forwards and backwards. Ask them to create patterns for others to solve, such as 112, 117, 122, 127 ... Ask: What is being added each time? When students are adding tens, relate this to counting on or back by tens to solve addition and subtraction problems.

Big Numbers

Have students work with a partner to enter the biggest number they can into the calculator. Ask them to read out their numbers and say which digit they entered first. Ask: Can you make a bigger number by starting with a different digit?

Really Big Numbers

Ask students to write down and then read out a "really big number" (or later, a decimal) for their partner to enter into the calculator. Check the screen with the written version of the number and say why they might not be the same.

Memory

Have students use the memory button to keep a progressive score for a school game of Scrabble, particularly when working out doubles and triples. For example: 12 add double 18, key in 12, M+, 18 x 2, M+, RCM.

Order of Operations

Have students explore the order of operations by doing a series of operations. Start by finding an easy way to get the answer to practical questions such as: (7 x $4) + (2 x $6). Have students share and compare methods and then try doing the same example with a calculator. Ask: Is the answer the same for some people and different for others? Why? How can you use the memory button to store the first result while working out the second part?

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Experimenting

Ask students to experiment with the CE, C, +/−, ÷ and % buttons and say what effect each has on the numbers.

Calculator Division

Have students compare the results of calculator division with results found through sharing materials. For example: Share ten markers between four students. The calculator says 2.5 and the sharing says two each and two left over to share. Ask: How are these the same? What does .5 show half of? Compare other division examples, such as: Three children in a family shared $37. Kate used her calculator and entered 37 ÷ 3. Ask: Can you help her explain what her answer of 12.333333 means? How much will each child get? Why?

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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Interpreting Displays

Have students interpret calculator displays to answer problems. Present these problems, calculations and calculator displays of the answer. Have students work out what the answers must be in each case.

- 1 kg of cheese costs $8.75. How much will 2.25 kilograms cost? 8.75 x 2.25. The display reads 19.6875.
- How many 1.5 m lengths of plastic rope can be cut from a total length of 32 m? 32 ÷ 1.5. The display reads 21.333333.

Memory Functions

Have students use the M+, M– and RM functions. For example: Use an electrical goods catalogue to spend $5000. Start by keying in 5000 into the M+. As the items are "purchased", key the amount into the M–. (To buy two CD players at $129 each, key in 2 x 129.) See how much is left by pressing RM. Have students think of other situations where this function would be useful.

Constant Function

Ask students to investigate how the constant function works for different operations. For example: Key in 6 x 4 = = =. The numbers 24, 144 and 864 appear in the display. Ask: Is the calculator constantly multiplying the four or the six? Now enter 6 + 4 = = =. Ask: Is the calculator constantly adding four or six? Now try 12 – 2 and 12 ÷ 2. Ask: Which part of the number sentence is constant? How can you check? How would this information be useful if you wanted to practise your eight times table or make your calculator count by 0.4?

Complex Calculations 1

Ask students to work out the correct order to enter complex calculations. For example: Calculate the cost of a list of items like 16 large pieces of card at $1.36 each, 8 m of laminating at $3.50/m and $2.50 for binding. Ask one group of students to key in the calculation as they say it: 16 x $1.36 + 8 x $3.50 + $2.50. Another group records the answer to each bit: 16 x $1.36, 8 x $3.50 and then adds the bits. Ask: Which answer is correct? How can using the memory function be useful in this situation? Does this occur with all the calculators you use?

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Complex Calculations 2

Extend the previous *Complex Calculations* activity by presenting scenarios such as: Jamie recorded his number sentence on paper as (4 x $1.20) + (3 x $2.35). His calculator does not have the order of operations function. What would he have to enter into the calculator to get the correct answer? Show Jamie how this is done.

Change of Sign

Have students use calculators with a change of sign function (+/–) to calculate with positive and negative numbers. For example: Pairs of students take turns to throw a ten-sided number cube (See Appendix: Line Master 8) four times and record the numbers in a 2 x 2 array. Subtract the second number from the first in each row and then add the two results with the calculator to get your score. The lowest score wins. For instance, if the result from the top row was –3 and the result from the bottom row was seven, you would need to key in 3 then +/– then + 7 followed by =. Ask students to check their answers on a number line.

5	8	–3
10	3	7

Compare Answers

Encourage students to compare calculator answers to answers from pen-and-paper or mental strategies when multiplying large numbers. For example: The calculator answer to 40 000 000 x 4 is E 1.6000000; the actual answer is 160 000 000. Ask students to find other examples. Ask: What do you have to do to the E number to get the correct answer? Why do you think this happens? Does this occur with other calculators?

Rounding

Have students investigate how the calculator rounds numbers. Ask them to find the decimal for one-third by dividing 1 ÷ 3. Then reverse the operation and multiply the result by three. Ask: Why is the answer not one? Try 2 ÷ 3. Ask: Why is the answer .6666666 for some and .6666667 for others? Multiply by three—why is the answer not two? Does this happen with other divisions? Why? Does it happen with decimals?

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CASE STUDY 5

Sample Learning Activity: K-Grade 3—Counting Device, page 195

Key Understanding 9: To use a calculator well, we need to enter and interpret the information correctly and know about its functions.

Working Towards: The end of the Matching and Quantifying phases

ACTION AND REFLECTION

It was Term 1 and Mr. Faraday wanted his grade 1 students to begin to learn about calculator functions. Ashana gave him the opportunity he had been looking for. She came rushing to Mr. Faraday exclaiming excitedly, saying *Look! Look! The numbers keep changing. I pressed three, then I pressed this big one (+), and then this one (=) and look! It goes 3, 6, 9, 12 ... I think it's counting!* The rest of the students all wanted to know what she had done, so Mr. Faraday used Ashana's directions to draw the appropriate calculator key sequence on the board:



When the effects of the constant key had been referred to as "counting" by Ashana and the rest of the class, Mr. Faraday did not immediately notice that students understood "counting" in different ways.

There was great excitement as the students followed this sequence to make their calculator "count" for them as well. Peter was particularly excited and told the class that the equals button was called the "channel changer" because *you keep getting a different picture on the screen*. Mr. Faraday realized that he had been able to introduce the constant function to the students by building on their own discoveries.

The question "Will it work for fives, too?" created an opportunity for everyone to experiment with different numbers. *Look how far I can count!* said Luke, who was counting by tens and had reached 50. When Mr. Faraday asked him how far he thought he could go, Luke ventured: *A hundred?* His incredulous tone implied that he had made a most improbable suggestion. He later returned and reported that: *Getting to 100 was easy ... a cinch! Now I can count by 100s ... 100, 200, 300, 400 ...*

One group of students found that by entering 1 + , then = = = = , they produced a sequence that was the same as for "real" counting, such as *one*, *two*, *three*, *four* ... There was the sudden realization for these students that this was indeed "counting" and that the calculator really could be used to find out how many just by entering 1 + , and then = as they pointed at each object they counted.

Mr. Faraday began to see the different ways students used the word "count". For some "counting" meant making sequences of increasingly larger numbers; others recognized it as "counting by" particular numbers, or "skip counting". Mr.

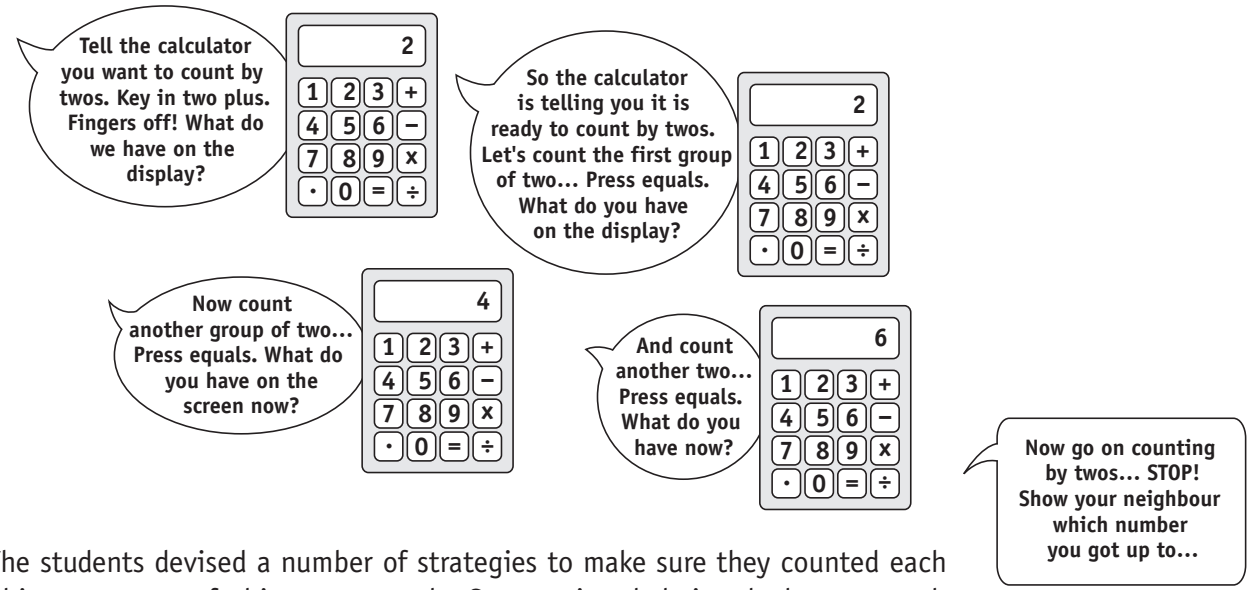
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Faraday soon realized that few saw any connection between the counting sequence they saw on the display and the process of counting to find out how many.

Mr. Faraday realized he would need to help many of the students make the connection between the number sequences created by using the constant function, and the process of finding out "how many" in collections. As he continued to focus students on learning to use the constant key efficiently, he frequently drew attention to this important understanding.

OPPORTUNITY TO LEARN

By the next day many of these young students had forgotten how to "make the calculator count". Mr. Faraday realized they would require practice if they were to use this important feature of the calculator effectively. Over the next few weeks, Mr. Faraday developed language patterns associated with using this function by repeatedly talking the students through the sequence in the following way:



The students devised a number of strategies to make sure they counted each object or group of objects once only. Some pointed their calculator at each one as they pressed the = key; others nodded their heads at the items as they carried out their counts.

DRAWING OUT THE MATHEMATICAL IDEA

While the students were working, Mr. Faraday constantly emphasized the connection between using this function and counting to find "how many". Therefore, he was surprised to hear Tiasa exclaim loudly, *Oh, I see! It's counting!* after she finished "counting" the glue bottles on the shelf using the function. On several previous occasions, Mr. Faraday saw her "count" other things with her calculator and assumed she already understood the process. Clearly, just showing her what to do was not enough—she needed the opportunity, time and experiences to make the connections for herself.

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Key Understanding 10

Thinking about what makes sense helps us to check and interpret the results of calculations.

Students should be expected routinely to check that their answers are reasonable. Above all, they should develop the expectation that the answers produced by their computation should make sense, and that if they do not, some correction is likely to be needed.

Checking Calculations

The first aspect of "making sense" involves checking on the likely correctness of any calculations undertaken. Computational errors are common and forgivable, but sticking with answers that *cannot* be right is nonsensical. Students should learn to check their answers using both their mathematical knowledge and their contextual knowledge.

Contextual Checks on Accuracy

Compare answers to what common sense suggests they should be. For example, students should notice that their answer cannot be right if the average height they have worked out is greater than that of anybody in the group.

Mathematical Checks on Accuracy

- ***Do the calculation a different way.*** Although we can check answers by going through the calculation a second time or re-entering the data into the calculator, we often reproduce the same error. Doing a calculation in a different order (adding up the column instead of down) or using a different mental strategy will be more likely to pick up a mistake.
- ***Estimate and approximate answers.*** Students should notice, for example, that their answer cannot be right if they have added three-quarters and two-thirds and get an answer less than one (such as five-sevenths), since both numbers are bigger than one-half.
- ***Think about the effect of particular types of operations.*** Students should notice, for example, that their answer cannot be right if they have multiplied by a number bigger than one and got a smaller answer, or multiplied by six and got an odd number.

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Interpreting Results of Calculations

The second aspect of "making sense" involves interpreting the results of calculations in sensible ways and deciding whether answers need to be rounded or adjusted in some way. For example, how we interpret remainders in division depends on the situation. It is more sensible to say that each student got seven stickers and there was one left over, than it is to say that each student got 7.2 stickers. Similarly, 90 eggs is 7.5 dozen or seven dozen and six left over. But the answer to the question "How many cartons are needed to hold all the eggs?" is eight, whereas the answer to the question "How many cartons can we fill?" is seven. For money, 4.5 on a calculator is read as $4.50 and $4.128 is rounded to $4.13 or possibly $4.15 or $4.10, depending on the context. Finally, it is generally not sensible to give answers that are more accurate than the data that produced them. Thus, if students' heights were found to the nearest centimetre, then finding an average more accurate than a centimetre usually will not be sensible.

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Numbers in Literature

Have students think about the numbers in literature and say if they make sense. At times change the numbers to focus them on the reasonableness of the numbers. For example:

- Have students make a sequel to *Counting On Frank* (Clement, 1990), using similar situations, such as Frank said, I accidentally knocked ten peas on the floor every day for a week so now there are hundreds of peas down there. Ask: Could this be right? Why? Why not?

Could It Be Right?

Invite students to imagine sharing out a quantity of things, or skip count, to select an answer that they think could be right. For example: Tell them that three students want to share 12 apples. Jim thinks they will get six each. Sharon thinks three and Hank isn't sure but he thinks they will get at least two each. Ask: What do you think?

Sensible Answers

Decide whether answers need to be rounded or adjusted in some way to make sensible solutions. For example:

- Dad needs half an apple to make muffins. How many apples does he need to buy? ("Half an apple" is not sensible.)
- Two families are going on a trip in two cars. There are four adults and five children. How many children in each car? ("2.5 children" is not sensible.)
- A bus can carry 20 children. How many buses will be needed for 30 children? ("$1\frac{1}{2}$ buses" is not sensible.)

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Problems in Context

Have students interpret answers given on the calculator within the context of the problem and write a description of what the result really means. For example: Dad gave me $45 to pay for lunch and buses for me and my five friends. How much is that each? Ask students to calculate 45 ÷ 6 to get 7.5. Ask: How much is that? How many dollars and how many cents? Vary to produce different numbers of decimal places.

Place Value

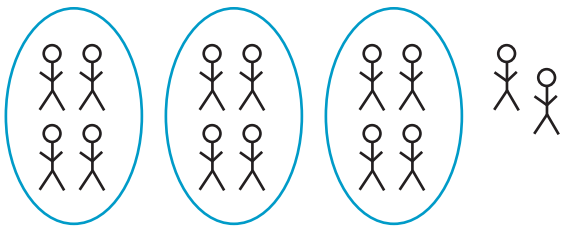
Ask students to use place value to think about the reasonableness of calculations. For example: Would five bags of potato chips be enough for our class of 27 and the 30 students next door? How would thinking about the number of students as "tens" and "ones" help?

Realistic Remainders

Encourage students to investigate ways of dealing with remainders in realistic situations. For example:

- Share seven litres of juice between three large containers for the class picnic.
- Five students share 18 marbles.
- Fourteen students need a ride to the T-ball game—how many cars will be needed if four can go in each car?

Discuss the sensible way to deal with the remainder in each situation.



Check the Answer

Have students check results of calculations carried out on calculators by checking the hundreds, tens and ones. For example: 420 + 346 = 866; 420 + 346 = 786; 420 + 346 = 760.

Sensible Answers

Invite students to choose a sensible answer to a problem, given a variety of answers. For example: If an experienced painter took three hours to paint a wall and the apprentice took seven hours, how long would they take to do the job together—21 hours, ten hours, four hours, three hours, two hours or half an hour? Ask: Which answer is the most sensible and why?

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Grades 3-5: ★ ★ Important Focus

Rounding

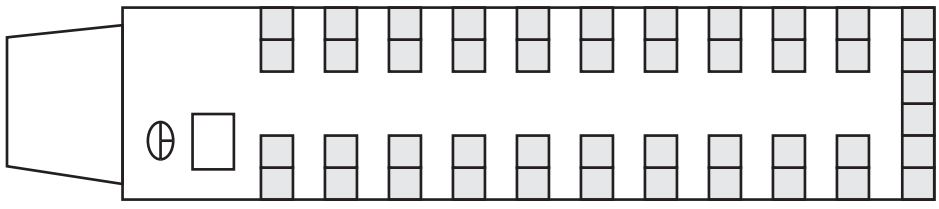
Have students round to make sense of calculator results. For example: How many rows of chairs will we need if we can fit 26 chairs across the assembly area and we want to seat 175 parents? Ask students to draw a diagram to represent the chairs and then write a division number sentence to use on the calculator to check their working out. Ask: What does the result of 6.7307692 tell us about the number of rows? How can you have .7307692 of a row? If we round this to 0.75, how much of a row should this be? Does your diagram show this exactly?

Age Estimates

Ask students to estimate how old they are in months, in weeks and then in days and then say which answer should be the biggest and by how much. Ask: How much bigger should the number of weeks be than the number of months? Does your estimate show this? Ask students to then use their calculator to find exact answers and compare their results with their estimates. Where there is a big difference, ask: Which answer is likely to be right? Why?

Bigger or Smaller

Have students anticipate whether the answers to problems are sensible. For example: Using a seating plan of a bus, ask students to find the total number of adults allowed to travel on the bus if there are two adults to each seat and six along the back. Then ask them to work out how many students can be seated if three students can take the place of two adults but, before they do, ask: Should the answer be bigger or smaller than the number of adults? Can the answer include fractions? Why not?



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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Problems in Context

Have students interpret calculator answers within the context of the problem and where necessary write a more appropriate answer. Present a series of problems and the calculator answer. For example: 1 kg of apples cost $1.99; how much will 2.5 kilograms cost? 1.99 x 2.5 = 4.975. Ask: Is the answer in kilograms or dollars? How many dollars and how many cents? If anyone suggests $13.75, ask: How do you know this answer is too big? What does the .975 mean? Draw out that it is .975 of a dollar, it is not the number of cents. Provide examples involving money, where the answers on the calculator may be, for example, 3.7 or 21.333333. Ask: How should we read these?

Spot the Mistakes

Without doing the calculation, have students explain why answers such as those below cannot be true, and what mistake the person might have made:

- Why can't 2.5 + 1.5 make 3.10?
- Why can't 0.5 x 0.5 make 2.5?
- Why can't 2.25 + 3.2 make 257?
- Why can't 4.5 x 100 make 4.500?

Real-life Remainders

Ask students to deal sensibly with remainders in real-life situations. For example:

- Lengths of ribbons measuring 80 cm are needed to hold each medal for sports winners. How many lengths can be cut from a 25-m roll of ribbon?
- A large 25-L container of punch is to be shared among the six groups in the class. How much does each group get?
- A farmer has 1324 horses to transport. A horse truck holds 235 horses. How many trucks will he need?

Sensible Answers

Have students interpret and label answers to calculations, so that the problem can be answered in a way that makes sense. Present students with the problems, the calculation and the answer. For example:

- If I travel at 95 km/h, how long will it take me to drive 250 km? 250 ÷ 95 = 2.63.
- I had my school photo enlarged. It was originally 120 cm long. Now it is 360 cm long. By how many times has it been enlarged? 360 ÷ 120 = 3.
- How many different outfits can I wear if I have three t-shirts and four pairs of shorts? 4 x 3 = 12.

Ask students to label the answer in each case so that it makes sense.

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Grades 5-8: ★ ★ Important Focus

Reasonable Answers

Without doing the calculations, have students say whether answers are reasonable or not. For example: Students mark the calculations done by a hypothetical student and say why some answers cannot be correct and explain what he or she has done wrong. Include whole number and decimal fraction examples:

238 + 26 + 412 = 910

$\frac{1}{2} + \frac{2}{3} = \frac{3}{5}$

475 ÷ 5 = 905

97 x 2.4 = 23.28

2.5 x 5.3 = 132.5

0.5 + 0.2 + 0.3 = 0.10

Making Problems

Ask students to think of problems that require the same calculation, but have different interpretations of the remainder. For example: 500 ÷ 45 can result in the answers 11.1111 or 11 remainder 5 or approximately 11. Ask students to invent number stories that would make each of these answers sensible.

Inaccuracies

A calculation gives an exact answer that may not always occur in real life. For example: An employee being paid at a rate of $15.20 an hour will receive $60.80 after four hours of work. However, sawing off four lengths of wood measuring exactly 15.2 cm each will require more than 60.8 cm of wood, because the saw cuts use up some wood. Ask students to brainstorm other measurement calculations where inaccuracies may occur in real life.

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Chapter 4

Patterns and Algebra

Investigate, generalize and reason about patterns in number, explaining and justifying conclusions reached.

Overall Description

Students observe regularities and differences and describe them mathematically. By identifying common features in mathematical situations, they are able to make generalizations about numbers, space and data. Thus they may observe that every time they combine three things with nine things and count, they get 12 things, and make the generalization that three add nine is always 12.

They know that there may be many patterns in the one situation, and generate and investigate a number of different conjectures about it. Students understand that a mathematical generalization must be true *always* rather than *mostly*, and that one exception invalidates it. They attempt to confirm or refute their own and others' generalizations and prepare arguments to convince themselves and others that a generalization *must* hold in every case and not only for all the cases tried.

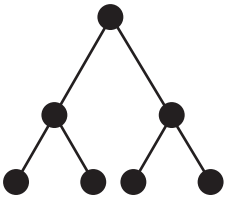
They write (and speak) mathematics clearly and precisely, expressing and explaining their generalizations verbally and with standard algebraic conventions.

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BACKGROUND NOTES

Finding the Rule for a Pattern

Students' work with patterns during their elementary school mathematics education should lay a foundation for their understanding of why we try to find patterns in things and how we go about it. They should investigate patterns in the familiar occurrences of their everyday lives, in their natural and human-made environment, and the more mathematically structured situations of the classroom.



Often, we have sufficient information to be able to decide what a pattern is exactly and make very reliable predictions from it. For example, since a virus replicates as shown left: we can work out what the exact number of a virus population is after 20 generations. However, at other times, we have incomplete information from which we try to make inferences. For example, we may have some information about whale numbers but not complete information. We nevertheless try to find a pattern or relationship that fits the data we have and use that pattern or relationship to make predictions about the future. Another way of saying this is that we generalize about the situation and apply the generalization to new instances.

Young students are usually not reluctant to generalize and suggest a rule that describes a pattern. The problem more often is that they generalize too quickly and do not test their rules against all the information. For example, given a sequence beginning 1, 2, 4, 7, ... they might see a doubling pattern for the first three terms and not check that it keeps working. This is not helped, of course, by textbooks and tests that expect students to continue sequences such as: 1, 2, 4, ... and accept only one possible extension. While we can hypothesize about what a pattern might be, given only the first three terms of a sequence, there is not enough information to say what it *must* be.

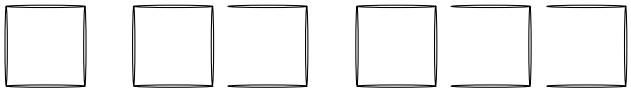
Sometimes more than one pattern will fit the information we have

Given no other information, the same short sequence of numbers could be the result of many patterns. For example, for a sequence beginning 1, 2, 3 to continue with 4, 5, 6 appears obvious, and it might well be that the person who began the pattern has the sequence of counting numbers in mind, but there are other possibilities. Thus the rule might be:

- begin with 1 and add one each time (that is, "count") — 1, 2, 3, 4, 5, 6, 7, 8 ...
- begin with 1, 2, then repeatedly add the previous two numbers — 1, 2, 3, 5, 8, 13, 21, 34 ...
- begin with 1, 2, 3 and repeat — 1, 2, 3, 1, 2, 3, 1, 2, ...
- begin with 1, 2, 3, 7 and repeat — 1, 2, 3, 7, 1, 2, 3, 7, ...

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Each rule fits the information we have; each is correct in that it fits the existing data. Thus, depending upon the rule, the next number could be 4, 5, 1, 7 or any other number! However, if we have additional information about the pattern, it may become clear that only one of the possible rules can be right. Suppose students were asked to use toothpicks to build a row of squares by adding on just enough toothpicks each time to add one square to the row:



At the first stage, there is just one square and it takes four toothpicks; at the second stage, there are two squares and it takes seven toothpicks; at the third stage, there are three squares and it takes ten toothpicks. The sequence for the number of toothpicks required at each stage begins 4, 7, 10, and students could conjecture that the rule connecting successive terms is "add three" and that the next number should be 13, and the next 16. In this case, they can check that their pattern rule works and even test it for the 100th diagram should they wish! We would probably not choose to check, since the diagrams explain *why* the pattern occurs and hence convince us that it is correct even for parts of the sequence we have not built.

The important point here is that, given ONLY the numbers 4, 7, 10, ... the best we can do is to find rules that fit the sequence and these rules might lead to any number of ways to continue. Any rule that fits the available information is correct and we cannot decide upon one of them for sure.

However, given more information about where the numbers 4, 7 and 10 came from and what the underlying principle was, we can decide for sure. That is, since we know how the toothpick sequence continues, we can work out how the number sequence must continue.

A pattern rule is right when it fits all the information we have and describes the pattern precisely

Where there is enough information to be able to work out for sure what the pattern must be, we say a rule is correct if it **must** work in every possible case. Where there is not enough information, we can still conjecture about what the rule is likely to be. Then we say a rule is correct if it fits all the information we have. In these cases, more than one pattern may fit the information we have.

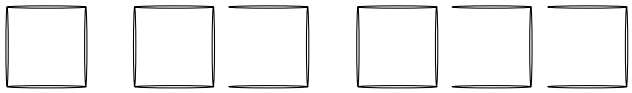
Either way, it is important that the rule we provide is precise. A rule that simply says "each number is double the one before" is not precise because it does not tell you where to start. Learning to describe a pattern precisely is not always easy, but there are some conventional

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mathematical ways to describe patterns which students should begin to use during Kindergarten to grade 3.

For example, in the toothpick pattern described on page 211, *physically* making a square with toothpicks, and then adding just enough toothpicks to make another square in the row, and then another could lead students to conclude: *You start by making a square with four toothpicks and then each time you add three toothpicks to make an extra joined-on square.* Such a description is quite precise and quite correct and it makes it fairly easy to get from one element in the sequence to the next (this is often called a "term").

However, using this description to continue the pattern to larger numbers is a little harder. While students may be able to suggest how many toothpicks it would take to make the tenth arrangement (the 10th term) or even the 100th, they would need to be careful to add on the right number of threes—a tricky business. This difficulty leads us to try to describe the relationship in a general way, by thinking of the 1st, 2nd, 3rd and 4th term in the sequence as having a position number (1, 2, 3, 4, ...) and linking the position number to the number of toothpicks needed:



Position 2 has one group of three extra toothpicks, position 3 has two groups of three extra toothpicks. So the number of toothpicks needed is four added to (one less than the position number) groups of three. So:

Number of toothpicks = 4 + (position number – 1) x 3

From this it is easy to see that the 100th term will be: 4 + 99 x 3 = 4 + 297 = 301!

Students should also learn to describe relationships that link pairs of quantities. For example, after playing "guess my rule"-type games, they should be able to state the rule clearly: *Whatever number we said, you halved it.* They should also learn to recognize the formulas that are used in daily life as examples of pattern rules. Thus, the formula for the area of a square, or for converting from Canadian dollars to American dollars, or for adding the right amount of water to the plant food are rules that link two quantities that "vary together".

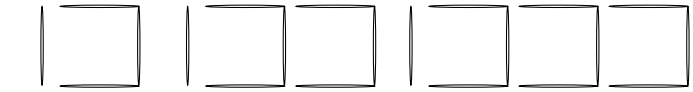
Two rules that sound/look different but produce exactly the same pattern are called equivalent (or equal)

Often when two people observe the same pattern, they will think about it differently. For example, two students each described this as a

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"doubling" pattern: 1, 2, 4, 8, ... However, one student suggested that you get each term by starting at one and then doubling the term before. The other student had looked at the differences between successive terms (also 1, 2, 4) and said, *You start at one, add one and after that the differences doubled each time.* These two rules sound different but they are mathematically equivalent or equal because they describe exactly the same pattern. After a discussion, these students could see why they were the same.

Sometimes thinking of a pattern in a different way helps us to write an equivalent rule that is easier to use. With the toothpick sequence on page 212, it is possible to "see" the arrangement of toothpicks like this:



So: *Number of toothpicks = 1 + position number x 3*
So, the 100th term will be 1 + 100 x 3 = 1 + 300 = 301.

These two rules are equivalent because they must produce exactly the same sequence of numbers.

Recognizing Common Types of Patterns

Students will get better at identifying patterns if they have sufficient and appropriate experiences in recognizing, producing and describing patterns. This experience needs to be as carefully planned as any other part of the mathematics curriculum to ensure that students experience a range of pattern types, learn to recognize common pattern types, and develop some of the strategies that are helpful for finding patterns. As students explore various patterns in number, their attention should be drawn to:

- the strategies that they found helpful in identifying patterns
- the similarities between certain patterns

Helpful strategies

It can be difficult to recognize a pattern immediately just by looking, and having some search strategies is essential. With sequences it often helps to test for common types. For example, in grades 2 to 5, students could test a sequence of numbers by finding the difference between successive terms and asking themselves: Is the difference constant? If not, is there a pattern in the differences?

Sequence	3		6		10		15 ...
Difference		3		4		5	

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Other search strategies for finding the pattern in a sequence of numbers include:

- Is there a constant ratio (or multiplier) between the successive terms? For example: 3, 6, 12, 24, ...
- Does doubling or halving work? For example: 36, 18, 9, 4.5, ...
- Is it the square numbers? Is it "almost" the square numbers? For example: 2, 5, 10, 17, 26, ...
- If there are fractions, what is happening to the denominators? Or to the numerators? Would using a common denominator help? For example: $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}$ is the same as $\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}$, ...

Similar sorts of strategies can be used in looking for patterns linking pairs of numbers. In this case, organizing information systematically will often help. For example, a student might find numbers that add to 20 by trial and error, and record them as they find them. Organizing the pairs vertically so that the first addend is increasing (see left) highlights the relationship between successive pairs. This should help the student see the pattern. Students should be assisted to organize their data in tables in ways that assist pattern-searching:

0 + 20
1 + 19
2 + 18
3 + 17

Position number	1	2	3	4	5
Number of blocks	1	3	6	10	15

Breadth	Height
24	1
12	2
8	3
6	4
4	6
3	8
2	12
1	24

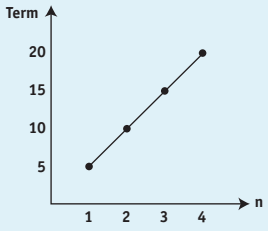
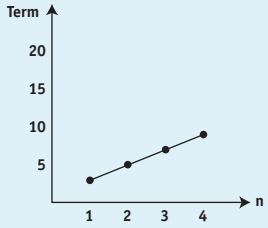
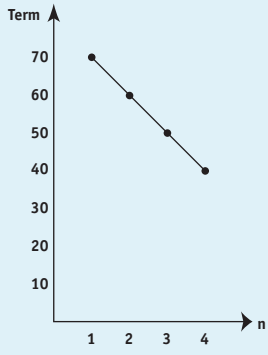
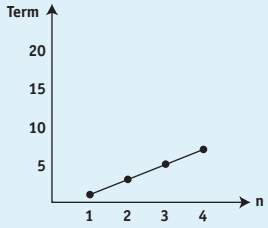
They need to learn not to focus only on the columns (or rows) of numbers but to look at the pairs themselves. Thus, in the table at left, focusing on the columns and looking for a pattern in the changes is not helpful. It is helpful to look at how the number pairs are related. (In this case, the product is always 24, that is, breadth x height = 24.)

Similarities Between Patterns

Students should also look for what is the same and what is different between various patterns, leading to simple classifications of patterns. During Kindergarten to grade 3, formal approaches to patterns are unnecessary, but students should put together sequences that involve constant addition/subtraction, such as "add three", and compare them with those that involve addition/subtraction by an increasing or decreasing amount ("first add three and then add one more than you did last time"). They could then comment on how the terms "grow" in each case. In the first example, the numbers get bigger at a steady rate, but in the second example, the numbers get bigger at an increasing rate as you go along. In grades 3 to 5, students could graph examples of each and compare the shape of the graphs. They may also learn to write general rules in progressively shortened forms and some may be ready to use letters to stand for variable quantities, although this should not be pushed.

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Constant Addition or Subtraction

Sequence	Rule to link terms	Table	General rule	Graph
5, 10, 15, 20, ...	start with 5 and add 5 each time	Position: 1, 2, 3, 4 — Term: 5, 10, 15, 20 (+5 each step)	the term is five times the number of its position = 5 x position no. = 5 x n (Note that 5 is the constant difference between the terms and it "shows up" in the rule multiplied by the positive number)	
3, 5, 7, 9, ...	start with 3 and add 2 each time	Position: 1, 2, 3, 4 — Term: 3, 5, 7, 9 (+2 each step)	the term is two times the number of its position then add 1 = (2 x position no.) + 1 = (2 x n) + 1 OR the term is 3 added to two times one less than the position number = 3 + 2 x (position no. – 1) = 3 + 2 x (n – 1)	
70, 60, 50, 40, ...	start with 70 and subtract 10 each time	Position: 1, 2, 3, 4 — Term: 70, 60, 50, 40 (–10 each step)	the term is 70 subtract 10 times one less than the position number = 70 – 10 x (position no. – 1) = 70 – 10 x (n – 1) OR the term is 80 subtract 10 times the position number = 80 – 10 x position no. = 80 – (10 x n)	
1, 3, 5, 7, ...	start with 1 and add 2 each time	Position: 1, 2, 3, 4 — Term: 1, 3, 5, 7 (+2 each step)	the term is two times one less than the position number and then add 1 = 2 x (position no. – 1) + 1 = 2 x (n – 1) + 1 OR the term is two times the position number and then subtract 1 = (2 x position no.) – 1 = 2 x n – 1	

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Students could draw a range of conclusions about the above patterns. For example, students in grades 3 to 5 might conclude that the four previous sequences are alike in the following ways:

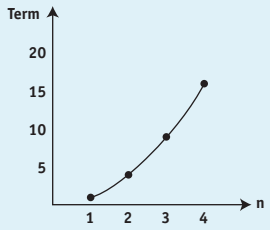
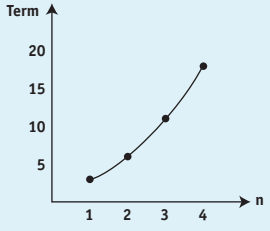
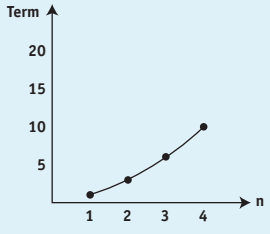
- to get the next term you add or subtract a constant amount to the term before
- the difference between two terms next to each other is always the same
- the numbers go up (or down) at a steady rate

In grades 5 to 8, they might add to this:

- the graph is always a straight line
- the constant difference shows up in the general rule, multiplied by the position number

and students might compare the alternative general rules to convince themselves that, although the rules look different, they say the same thing.

Adding or Subtracting a Constantly Increasing Amount

Sequence	Rule to link terms	Table	General rule	Graph
1, 4, 9, 16, ...	start with 1 and add 3 and add 2 more than that each time *The difference between the differences is constant (2 in this case).*	Position | Term 1 | 1 + 3 2 | 4 + 5 3 | 9 + 7 4 | 16	the term is the square of the position number = n x n *These are the square numbers.*	
3, 6, 11, 18, ...	start with 3 and add 3 and then add 2 more than that each time *The difference between the differences is constant (again, 2).*	Position | Term 1 | 3 + 3 2 | 6 + 5 3 | 11 + 7 4 | 18	the term is the square of the position number add 2 = n x n + 2	
1, 3, 6, 10, ...	start with 1 and add 2 and add 1 more than that each time *The difference between the differences is constant (1).*	Position | Term 1 | 1 + 2 2 | 3 + 3 3 | 6 + 4 4 | 10	the term is the position number times one more than the position number, divided by 2 $= \frac{n \times (n + 1)}{2}$ *These are the triangular numbers.*	

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In grades 3 to 5, students might conclude that the three sequences above are alike in the following ways:

- to get the next term you add or subtract an amount that changes each time, but the change is fixed
- the numbers go up (or down) more quickly (or slowly) as you go along

In grades 5 to 8, they might add to this:

- the graph is always a curve that increases or decreases at an increasing or decreasing rate
- the general rule always has a n x n in it (while the square numbers themselves will be accessible to students, the more complex rules involving squares may be too challenging to express symbolically)

Similar approaches to other types of patterns can also be taken. Pattern types should at least include those involving:

- multiplication or division by a constant amount (3, 6, 12, 24, ...)
- reciprocals ($\frac{2}{1}$, $\frac{2}{2}$, $\frac{2}{3}$, $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$ ...)
- cycles or repeats (2, 4, 6, 8, 2, 4, 6, 8, ...)

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Patterns and Algebra:
Key Understandings Overview

Teachers will need to plan learning experiences that focus on the following Key Understandings (KU). These Key Understandings underpin achievement of this family of concepts. The learning experiences should connect to students' current knowledge and understanding rather than to their grade level.

Key Understanding		Description
KU1	We use regularity or pattern to infer one thing from another thing and to make predictions.	page 220
KU2	Representing aspects of a situation with numbers can make it easier to see patterns in the situation.	page 232
KU3	To describe a number pattern means to provide a precise rule that produces the pattern.	page 244
KU4	There are strategies that help us become better at recognizing common types of patterns.	page 254
KU5	Our numeration system has a lot of specially built-in patterns that make working with numbers easier.	page 262
KU6	Some numbers have interesting or useful properties. Investigating the patterns in these special numbers can help us to understand them better.	page 270

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Grade Levels—Degree of Emphasis			Sample Learning Activities
K-3	3-5	5-8	
★ ★ ★	★ ★	★ ★	K-Grade 3, page 222 Grades 3-5, page 224 Grades 5-8, page 226
★ ★	★ ★ ★	★ ★ ★	K-Grade 3, page 234 Grades 3-5, page 236 Grades 5-8, page 238
★	★ ★	★ ★ ★	K-Grade 3, page 246 Grades 3-5, page 248 Grades 5-8, page 250
★	★ ★ ★	★ ★ ★	K-Grade 3, page 255 Grades 3-5, page 257 Grades 5-8, page 259
★ ★	★ ★ ★	★ ★ ★	K-Grade 3, page 264 Grades 3-5, page 266 Grades 5-8, page 268
★	★ ★	★ ★	K-Grade 3, page 272 Grades 3-5, page 274 Grades 5-8, page 276

Key

★ ★ ★ **Major Focus**
The development of this Key Understanding is a major focus of planned activities.

★ ★ **Important Focus**
The development of this Key Understanding is an important focus of planned activities.

★ **Introduction, Consolidation or Extension**
Some activities may be planned to introduce this Key Understanding, to consolidate it, or to extend its application. The idea may also arise incidentally in conversations and routines that occur in the classroom.

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Key Understanding 1

We use regularity or pattern to infer one thing from another thing and to make predictions.

Students are most likely to recognize the significance of pattern when their attention is regularly drawn to it in the context of other work across the curriculum, in all strands of mathematics and in conjunction with the other Key Understandings for Patterns and Algebra. As students note and use familiar patterns in their everyday lives and the more structured patterns of the classroom, their attention should be repeatedly drawn to the following two aspects of this Key Understanding.

Firstly, in a mathematical context, we use the word "pattern" to refer to the underlying regularity or ongoing repetition in a situation. For example, in a fabric design made by turning and sliding a motif, it is not the motif or the overall design that is the "pattern", rather the pattern is the regularity of the turning and sliding. To describe the pattern means to describe *how to get* the design or overall picture. To copy a pattern means to reproduce the regularity.

Secondly, the main reason we focus upon pattern is that patterns enable us to predict, expect and plan. Mostly, we respond to patterns without noticing that we have. Thus, even though the details of each day are different, the underlying regularity or pattern in our days enables us to infer the time from signs around us, such as the car noises and voices outside school signal that the school day is almost over. Pattern also helps us to predict and hence plan, such as when buses are scheduled to fit typical days.

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Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ use obvious patterns to make copying arrangements and sequences easier; that is, they no longer reproduce simply by matching each component one at a time ■ are able to talk about the regularity in the events of their own daily life using such simple language as "pattern," "over and over," "repeat," and "again" ■ are beginning to understand that it is the regularities in everyday life that enable them to "know what to do" at various times and to say what is likely to happen in the future

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Sample Learning Activities

K-Grade 3: ★ ★ ★ Major Focus

Incidental Activities

For example:

- ***Learning a song.*** Ask: How did you manage to learn the words so quickly? What is the pattern? How did it help?
- ***School's out.*** Say: Some of you are getting ready to go. How come? Why do you think it is time?
- ***Stories.*** When telling and retelling stories with repetitive features, stop and ask students to predict what comes next. Ask: How did you guess (remember) so well?

Daily Activities

Encourage students to look for the repeating events in daily routines, such as arriving at or leaving school. Ask them to draw the separate events. Make copies of the drawings for them to set out, showing the events over several days. Encourage students to say what happens over and over again, such as come, go, come, go OR come, put bag away, set up, bell goes, come, put bag away, set up, bell goes.

Sound and Movement

Have students follow sequences of movements and sounds, comparing those that are patterned with those that are not. For example: Ask them to first copy and join in with a "clap clap, turn around" repeated sequence and say what they think comes next. Then copy and join in. Where the sequence has no obvious repetition, stop the sequence and ask: What comes next? (They can't tell.) Which was easiest to follow—the first group of movements or the last one? Why?

Lining Up

Invite students to use the repeating pattern in the actions of a line of students to predict what comes next and what they will have to do. For example: Ask students to form a line around the room. Walk along the line and as you tap each student on the shoulder tell them to "stand, stand, sit, sit, sit, sit, stand, stand, sit, sit, sit, sit", and so on. Say it as a chant and have students join in as you go. Once the rhythm is established, stop and ask: What will Kia (next student) be doing? Repeat. Extend this to predicting several ahead. Ask: How do you know? (Because the teacher is not just making it up as she or he goes along—there is a pattern.) Stop giving the instructions and ask the rest of the students, one at a time, to sit or stand according to the pattern.

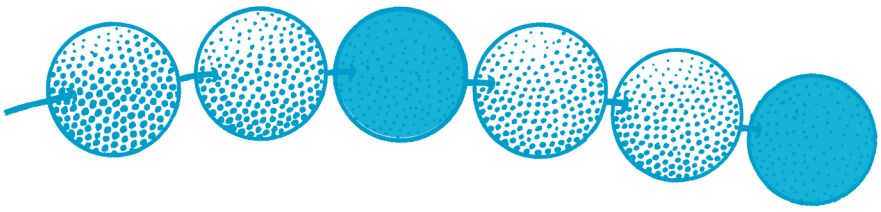
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Watching the Lineup

Have students look at a line of students pre-arranged in a repeating pattern such as in the previous *Lining Up* activity. Ask them to decide what the pattern is and what the next several students will need to do to continue the pattern. Focus on the repeating parts.

Necklaces

Ask students to reproduce a string of beads that shows a repeating pattern, such as white, white, blue, white, white, blue. Draw out the "pattern" by having students chant aloud the sequence of colours. If necessary, model the rhythm. Ask students to make their string longer than the original. Ask: How do you know what comes next? Repeat for a variety of repeating patterns and extend to three colours.



Varied Objects

Have students repeat the previous activity, but include several types of objects or beads where the only repetition is in the colour. For example: The repeating pattern might be white, white, blue, white, white, blue ... but the actual objects quite varied. Draw out that this pattern is in the colour. Other things could be different. Repeat with a line of students: girl, girl, boy ... Each student is different, but there is a pattern in the gender. Repeat for eye colour, ignoring other variations.

Decades

Count in rhythm with students as they go up, over and through the decade numbers. Use variations in pitch and volume of voice to emphasize one to nine repeating within each decade, and the decades (tens) also following this pattern. Stop periodically and ask: How do you know what comes next? What is the pattern? How does it help?

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Incidental Activities

See also Incidental Activities, page 222.

- ***Learning a dance.*** Ask: What makes remembering the steps easier? How does the pattern help?
- ***Checking work.*** Ask: How could you tell you must have made a mistake? How did the pattern help you decide where you had gone wrong? (See Case Study 1, page 228.)

Dance Patterns

Have students learn the steps to a simple dance, such as the heel-toe polka, and then identify and write down the series of steps that makes the basic repeating unit of the dance. Ask students to then make their own dance by changing this basic repeating unit in some way, and teach it to other students.

Daily Activities

Have students record their own daily events on a 24-hour timeline for five to ten weekdays. Ask them to line up their timelines and look for regularities. Ask: Do you have a regular time for getting up? For going to bed? What else can you say about a typical day? What might cause a change in the pattern of your days? Have students write a letter to an e-mail pal from another country, describing what their own typical school day is like. Compare these with similar responses from e-mail pals.

Families

Extend *Daily Activities*, above, by having students compare each other's days. Ask: What is the same? What is different? Which parts of the day are most alike? Is there a pattern that fits most students? Draw out that even though each day is different and each family is different, there are patterns in our days that result from natural (dark and light) and designed (school timetable) processes, which make generalizing possible and sensible.

Shadows

Have students graph the length and direction of the shadow of a plant at different parts of the day over a number of days. They can then use this to predict which part of the playground will be the shadiest during recess and lunchtime. Ask: Is there likely to be more shade at recess or lunchtime tomorrow? Will this be the same every day? How do you know?

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What Time Is It?

Pose simple problems involving time. For example: It is 9 o'clock now, what time will it be in two hours? In three hours? In four hours? Why not 13 o'clock? How do you know? (The hours of the day have a repeating pattern: 1, 2, 3, 4 ... 12, over and over.) Challenge students further. For example: If the bus leaves at 9:30 and the trip takes two hours, what time will we arrive? What if the bus breaks down and the trip takes three hours? We can predict because of the patterns built into our system for describing the time of day (which match the natural cycle of day and night).

More Time

Extend *What Time Is It?*, above, to the repeating or cyclic patterns within each hour: the quarter hours (o'clock, quarter past, half past, quarter to) and the minutes (cycles through 60 minutes, for each hour).

The Answer Is ...

Ask students to make up an addition that has a specified answer, such as 24. Ask them to make up another, and another—as many as they can. Encourage them to put their sums in order and fill in any gaps. Ask: Can you use patterns in the numbers to make sure you have all the pairs of whole numbers that add to 24?

$$1 + 23 = 24$$
$$2 + 22 = 24$$
$$3 + 21 = 24$$

Ask: What happens to the second number as the first goes up by ones? Have students explain to a partner. Challenge them: I worked out that 237 + 492 = 729. Can you find some other pairs that must add to 729 (without doing the calculation)? If I increase the 237 to 238, what do I need to do to the 492? (See Case Study 1, page 228)

More Answers

Repeat the *The Answer Is...* activity above, but using subtractions rather than additions to give a specified number.

Bus Times

Have students investigate the times on bus timetables to work out the regularity. Ask: How often do buses go on weekdays? How often on weekends? If you miss the bus, how long will you have to wait until the next bus arrives on a weekday? On a weekend?

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Sample Learning Activities

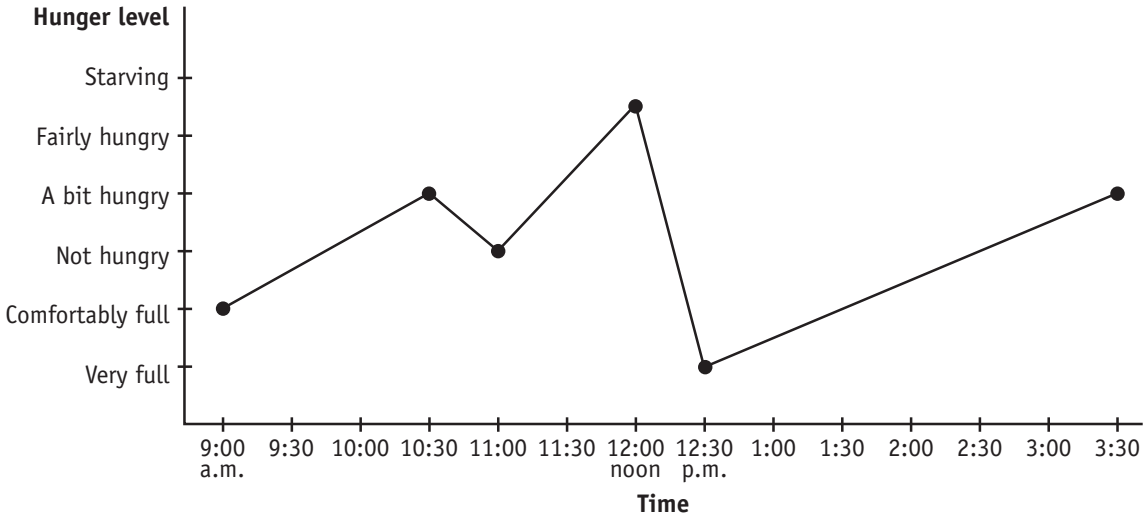
Grades 5-8: ★ ★ Important Focus

Incidental Activities

- Students making a wall chart to show π (pi) to 30 places (3.14159265358979323846264338327 9) find it hard to quickly copy it accurately. Compare with copying a recurring decimal such as 0.231231 ... Ask: Why is this?

How Hungry?

Have students record and then graph their hunger level every half hour through a typical day. Ask: How does your graph compare to others? When does hunger suddenly drop? How long does it take to drop? When is it least? Highest? How long does it take to rise to its highest? Does your graph show what happens? Discuss whether the differences between students' graphs reflect real differences in their days (how regularly they eat and when they eat). Have students re-draft graphs to better show the typical hunger pattern in their own days.



Other Features

Repeat the previous activity for other variable features of students' days, such as how temperature varies, how active they are, or how tired they are feeling through the day.

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Holidays

Ask students to compare weather patterns in different locations to predict what weather will be like and to plan a trip. For example: Compare weather data for St. John's, Toronto and Vancouver and consider questions such as: At what time of year would you most like to go to St. John's? Toronto? Vancouver? If you go to St. John's in July, what sorts of clothes should you take? What if you go to Toronto? Draw out that even though each day (year) is different, the general weather pattern enables you to plan with some confidence. Extend this to investigate how weather patterns affect crops, sea life, population (related to tourism) and special events.

Investigating Primes

Give students a three- or four-digit number that you know is a multiple of seven, such as 2247. Ask them to decide whether or not it is. After they have confirmed it is, ask: What is the next multiple of seven? And the next? How do you know? Then give students a number you know to be prime, such as 2331, and challenge them to decide whether or not it is. After they have confirmed it is, ask: What is the next prime? What is the problem? (No pattern, cannot predict.)

Copying Decimals

Write on the board two lists of ten numbers, each with, say, 12 decimal places:

- On List 1, include only numbers in which the digits are obviously patterned, such as 3.454545454545.
- On List 2, include only numbers in which digits are not obviously patterned, such as 3.414283371429.

Ask one student in each pair to copy List 1 as quickly (but correctly) as they can, and the other to copy List 2, and to record which partner finishes first. Collect data as a class. It is likely that more students copying List 1 will finish first. Discuss why this might be. Ask: How did the pattern help? Link to what sorts of phone numbers are easiest to remember. (This could precede work on recurring decimals.)

Calculating with Patterns

Have students use patterns to make calculating easier. Present this scenario: Sam was doing homework. He drew diagrams to work out $\frac{1}{4} \times \frac{1}{2}$, $\frac{1}{3} \times \frac{1}{5}$ and so on. He thought $\frac{3}{4} \times \frac{6}{7}$, $\frac{2}{3} \times \frac{5}{8}$ and $\frac{5}{6} \times \frac{1}{4}$ were too hard. He had a calculator that could calculate fractions so he decided to use that. He worked out that $\frac{3}{4} \times \frac{6}{7} = \frac{18}{28}$, $\frac{2}{3} \times \frac{5}{8} = \frac{10}{24}$. Ask: Can you see a pattern that would help Sam work out similar examples without using his calculator?

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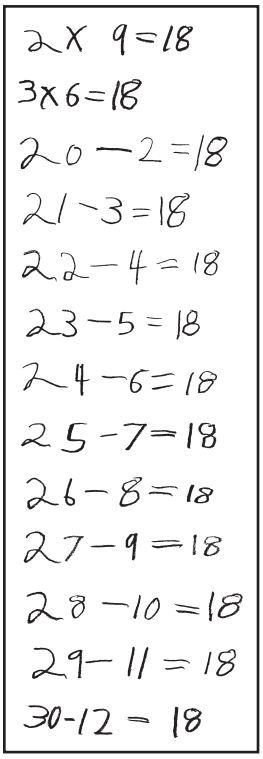
CASE STUDY 1

Sample Learning Activity: Grades 3-5—The Answer Is …, page 225

Key Understanding 1: We use regularity or pattern to infer one thing from another thing and to make predictions.

Working Towards: The end of the Quantifying and Partitioning phases

As Ms. Maxwell walked past a class, she heard a boy say "18". The teacher said, "That's right!" What question could the teacher have asked? Write down as many as you like.



BACKGROUND TO THE SAMPLE LESSON

Ms. Maxwell originally set the students in her grade 3 class a task designed to help her understand what they knew about numbers and number patterns. (The task is shown at the left of the page.) The students had constant access to calculators and were encouraged to use them as a matter of course.

Some students produced a quite varied list of examples, others lists of related examples. Some also experimented with their calculators to produce expressions that were well beyond what one would normally expect of students in grade 3. Almost all produced many more calculations and questions than Ms. Maxwell would have set them! Over the next few weeks, she returned regularly to selected work samples as a starting point for different activities.

Several students relied heavily on patterns to generate a large number of related calculations. Ms. Maxwell took the opportunity to use these to develop the key understanding that we can use pattern to infer one thing from another thing and to make predictions.

SETTING UP THE SITUATION

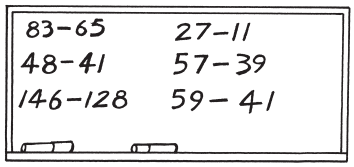
Ms. Maxwell made an overhead of Maria's work and showed it to the whole class. Ms. Maxwell asked Maria to explain what she had done. She immediately stated that she could see she would get the same answer for 22 – 4 as for 21 – 3 and for 20 – 2, and so she was able to make up a lot of examples very quickly by adding one to each number.

Ms. Maxwell then asked all the students to think about what Maria had done, to talk it over with their partner and decide whether it would always work. She called on several students to say whether they would be prepared to rely on the pattern Maria had used and why. Because all of Maria's examples worked, the students said they would accept it. Ms. Maxwell then asked them how far they would be prepared to go. Would they go to higher numbers? Would they add 100 to each number? She wanted to force them to be explicit about what rule they were using.

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DRAWING OUT THE MATHEMATICAL IDEA

Ms. Maxwell then returned to the key point she wanted to draw out by removing Maria's overhead and asking students to check whether the following six calculations had an answer of 18.



They were to put the pens/calculators down and sit up straight once they had decided. Ms. Maxwell did not immediately review their decisions but rather noted with simulated surprise that it had taken them quite a while to decide, and yet Maria had produced lots of calculation more quickly. How come?

Maria was bursting to explain that, once she began to use the pattern, she did not need to check each one. Ms. Maxwell asked her what was different in her list and the list above that made it easier for her. After a pause, she (and others) said that her list was in order and so she could just go up one at a time. She could decide that a calculation would have to give 18 without having to work each one out. Although she did not use the word, she was saying that she could use one difference to infer a lot of others.

Other students generally agreed that she had not needed to check each one, although a few seemed to think that relying on the pattern somehow minimized her achievement. Ms. Maxwell did not overtly challenge this view that "harder is better", but instead made a big point of saying how in mathematics we always try to find the easiest way to solve problems; that the reason we look for patterns in math is so we do not have to do each problem as though it were new. We can use what we learn in one problem to do others more easily.

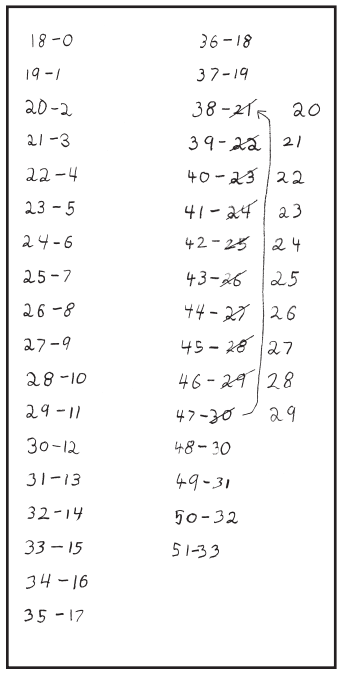
EXTENDING THE MATHEMATICAL IDEA

To Ms. Maxwell's delight, one of the students then asked: *What if you make a mistake in one? Wouldn't all the rest be wrong?* Had he not asked this she was prepared to bring it up herself. Ms. Maxwell turned to Andre and said: *Andre can tell you about that, can't you?* She pulled out the worksheet he had produced and Andre explained that he had used a pattern to produce his list of calculations, but suddenly realized when he got to 47 – 30 that it could not be right. He said: *I knew I must have made a mistake, so I just went back up the list until I found where I went wrong.*

Ms. Maxwell asked: *How did you do that? Did you work out every one?*

No, he said. *I looked at the pattern and could see that the first number changed by one each time, but the second number changed by two here* (pointing to where the error occurred).

So what is the lesson here? Ms. Maxwell asked.



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In the brief discussion that followed, Ms. Maxwell was able to draw from the students that we can make mistakes even when using a pattern. Therefore, we need to check every now and again that we have not gone wrong—preferably, as Andre did, with examples that are easy to check. However, organizing the examples in a list so that the pattern in the numbers was obvious helped Andre to notice and find his mistake.

THE NEXT PHASE

Ms. Maxwell then moved on to develop two other Key Understandings:

- Firstly, our numeration system has a lot of specially built-in patterns that make working with numbers easier (Key Understanding 5, page 262).
- Secondly, there are strategies we can practise to help us do calculations in our head (Computations, Key Understanding 5, page 152).

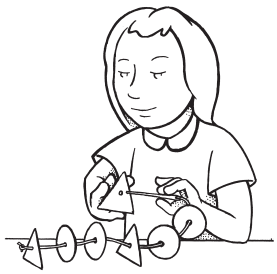
Thus, students learned to use related calculations, such as: 56 – 37 is the same as 59 – 40, which is much easier.

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Did You Know?

It is important to check that students actually do recognize and respond to familiar or readily observed regularities in mathematical situations. Sometimes we think they are copying a pattern, when they have not even noticed it.

For example, students might be asked to copy the pattern in a necklace of threaded shapes: ●●▲●●▲●●▲●●▲. If students copy the necklace by carefully matching each shape one at a time, they have either not noticed the pattern or are not seeing its relevance to the task. Either way, they have not copied the **pattern**. For these students, the order of shapes may as well have been random.



When students recognize the pattern, they can use it when copying the necklace, perhaps muttering "circle, circle, triangle" over and over again. They can reproduce the sequence without looking back constantly and they notice errors and can self-correct the copy. Also, once they see the pattern, they will be able to continue it beyond the items provided to extend the necklace.

Many students will do this quite naturally, but some will not. They can be helped by the teacher modelling, through voice and actions, the rhythm of the pattern.

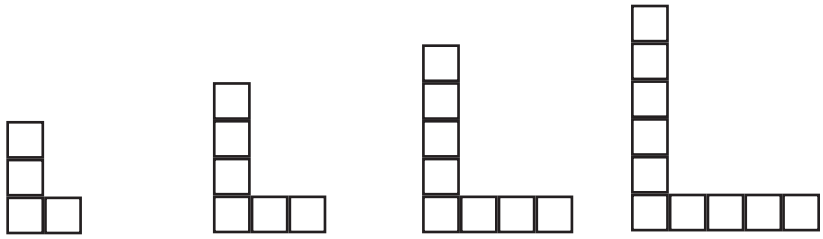
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Key Understanding 2

Representing aspects of a situation with numbers can make it easier to see patterns in the situation.

Students should learn that representing aspects of a situation with numbers and then looking for patterns in the numbers can help us understand the situation better, often making patterns more obvious and predictions easier. For example, in Kindergarten and grade 1, students observing a necklace that shows repetitions of "bead, leaf, leaf, shell, shell, shell" might chant the numbers *1, 2, 4, 1, 2, 4, 1, 2, 4, 1, ...* and then say, *It goes 1, 2, 4 over and over*, or *It goes one bead, two leaves, four shells over and over*. From the number pattern, students could ask and answer questions, such as: *If I keep making the necklace bigger will I need more beads, more leaves or more shells? If I use 16 shells, how many leaves will I need?*

In grades 2 to 4, students might be asked to build a sequence of L shapes with squares, starting with the one shown below, so that the arm lengths increase by one each time.



They could represent the pattern made by the number of squares needed as: 4, 6, 8, 10, ... and use this to predict how many squares they will need for the next L and the next. Using the constant addition function on their calculator, they might predict how many squares it would take to make the tenth or the 20th "L".

In grades 4 to 6, students could generalize further and find a way to say how many for the 100th L, without having to count through the whole 100 terms. They could also work out whether you could build an L with 127 squares. This is not obvious just by looking at the Ls. All the early numbers, however, are even and each new L involves adding two, so 127 cannot work.

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Students should also learn that the same number pattern can be present in many different situations. For example, the repeating pattern 1, 2, 4, 1, 2, 4, ... in the necklace above is the same as in "one squat, two jumps and four claps" repeated over and over. This is useful because it means that the questions we have answered about the necklace have also been answered about the matching body actions.

Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ can represent a simple repeating or counting pattern using numbers, and use the numbers to help them continue the pattern ■ will show the same repeating or counting number pattern in different forms and media
Quantifying	■ can represent a variety of patterns using numbers, and will find ways to represent number sequences in materials in ways that help to expose the pattern
Partitioning	■ build sequences of simple shapes (squares, triangles, Ls) that change systematically, and write a matching number sequence
Factoring	■ understand that, for example, the same sequence of shapes could be represented with different number sequences, depending on which aspect of the sequence of shapes was focused on

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Sample Learning Activities

K-Grade 3: ★ ★ Important Focus

Necklaces

Ask students to thread two colours of beads on string in a repeating pattern, such as “blue, blue, red, blue, blue, red” and record as a number sequence: 2, 1, 2, 1, 2, 1, ... Later, extend to more than two parts repeating.

Isolating the Pattern

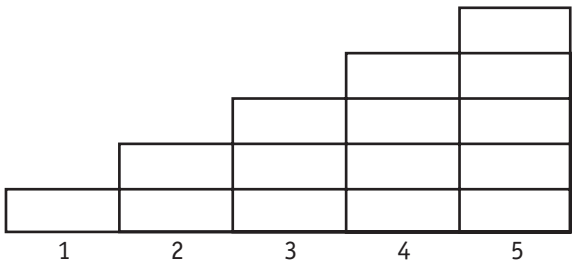
Have students isolate the repeating section of a sequence. For example: Draw boxes around two blue, one red in the previous activity. Ask: What is the shortest way to say the pattern? What is the smallest part that is repeated?

Lining Up

Invite students to explore a repeating pattern in a lineup of students. Ask students to lineup. Walk along the line and tap each student on the shoulder, telling them to "stand, stand, sit, sit, sit, sit, stand, stand, sit, sit, sit, sit," and so on. Say it as a chant and have students join in. Then have students use numbers to describe the repeating pattern in the lineup. Model the process. Say: It was “stand, stand, sit, sit, sit, sit, repeated over and over”. So that is two stands, four sits, over and over, or 2, 4, 2, 4, 2, 4, 2, 4, 2, ... Ask students to use the same pattern to make other arrangements, such as “two twirl and four stamps” repeatedly.

Stairs

Have students make a staircase out of coloured paper, felt or magnetic squares. Ask them to record the number pattern for their staircase, such as 1, 2, 3, 4, 5 or 3, 6, 9.



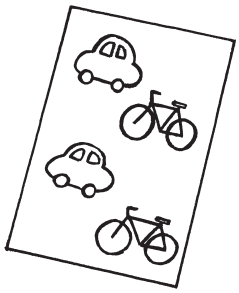
Print-making

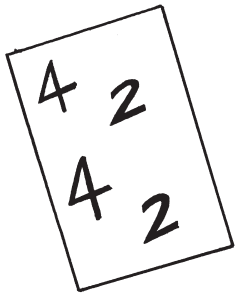
Focus on the number pattern in students’ sequences. For example, when print-making or stamping, a student may make ●●●●●●●●●●. They could name it as a 4, 2, 4, 2, 4, 2 pattern and look for 4, 2, 4, 2 patterns made by others. Focus on the idea that they are the same pattern.

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Matching Patterns

Play games involving matching number patterns to cards with object patterns. Make several object pattern cards for each number pattern. Ask students to find several object patterns for the same number pattern. For example: 4, 2, 4, 2, 4, 2 would match the pattern of wheels of a car, bike, car, bike, and so on. or the legs of a dog, hen, dog, hen. Ask: Why can there be many sets of picture patterns for each number pattern?





Sound and Action

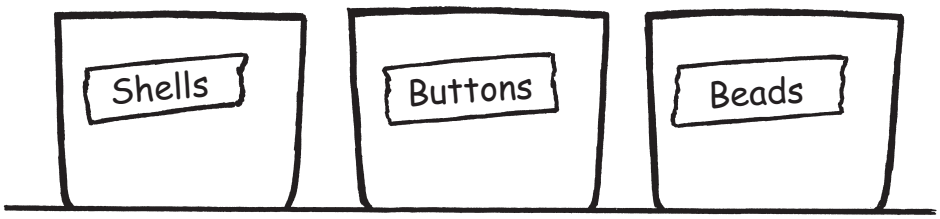
Invite students to create sound or action patterns for others to follow, based on a single number pattern. For example: Make instructions for a line dance using four stomps, eight claps, four kicks, eight skips. Copy and continue the pattern, then create other steps in a 4, 8, 4, 8 pattern. Students say whether each sound or action pattern follows the given number pattern.

Reference Chart

Make a chart of the number patterns generated during activities for students to refer to when using patterns to create art or dances and working on the How Many? activities below and on page 247.

How Many?

Ask students to use a number pattern to plan how many objects they will need to make a given number of repeats. Ask them to choose a number pattern from the class chart made previously, such as 1, 4, 2, and choose from three containers where there are twice as many shells as buttons and beads to make a pattern. Help them plan how many of each object they will need if they make three repeats.



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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Material Patterns

Ask students to choose materials to create a sequence that "shows" a provided number sequence, such as 3, 4, 2, 3, 4, 2, 3, 4, 2, ... (♣♣♣❝❝❝❝○○ ♣♣♣❝❝❝❝○○...). Compare displays and draw out that all the displays share the same pattern. Have students then work in groups to use the pattern in other "creative" ways, such as body actions and poses, music, dance steps, and people.

Pasta Patterns

Have students plan a pattern using three types of pasta, such as three spiral, two bow and one shell. After constructing the pattern, ask students to identify and group similar patterns and write the number sequence for each group. Ask: Does a 1, 4, 1 pattern look different from a 1, 1, 4 pattern? Why or why not?

Tricycles

Pose the following scenario: You need to help a tricycle manufacturer work out how many parts are needed for different-sized orders. Begin with wheels. How many wheels will be needed for an order of one tricycle, two tricycles, three tricycles? How many wheels will be needed for nine tricycles? Ask students to write a number pattern to help find the answer. Extend this to produce a table showing the number of parts for different numbers of tricycles.

Number of tricycles:	1	2	3	4
Number of wheels:	3	6	9	12
Number of seats:	1	2	3	4
Number of hand grips:	2	4	6	8
Number of tires:	3	6	9	12

Draw out that the same number pattern may apply to different parts. For example, if you know how many wheels, you also know how many tires.

Letter Grid

Have students write their name on a grid, noting one letter per square. They can choose the size of the grid, between 4 x 4 and 8 x 8. At the end of each line, the letters are continued onto the next line until the whole grid is filled. Ask students to find others with the same pattern and try to decide why their patterns are the same. Is it the number of letters or the number of letters and the number of squares in the grid that is the same?

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They then write a number pattern to explain their pattern. For example: For the name Kate in a 5 x 5 grid, the pattern would be 4, 1; 3, 2; 2, 3; 1, 4.

K	A	T	E	K
A	T	E	K	A
T	E	K	A	T
E	K	A	T	E
K	A	T	E	K

Triangular Numbers

Investigate different arrangements for the sequence of triangular numbers: 1, 3, 6, 10. For example: How many line segments are needed to join 2 dots, 3 dots, 4 dots, 5 dots ...? How many blocks are needed to make a staircase 1 block high, 2 blocks high, 3 blocks high ...? How many triangles can be found in a triangle with a fold from 1 of its vertices, 2 folds, 3 folds ...? Ask: How are all of these patterns the same? Why do you think this sequence is called triangular numbers?

What's Next?

Have pairs of students make a model of the sequence 1, 2, 3, 5, 8, ... with toothpicks, and then say what the next three numbers might be. Ask students to use their model to explain to others how they decided what the numbers should be. Ask: How did you know to put 13 toothpicks next?

Area Problems

When studying area, have students find the possible dimensions for a rectangular patio of 24 squares. Order the results to confirm that all possibilities have been discovered. Ask: Is there a pattern in the numbers for length and width? What is the longest and shortest perimeter possible for this shape? Have students use the number pattern to help find the longest and shortest perimeters of other sized patios, without having to make the model.

Puppies

Have students look for adding and subtracting patterns to make predictions about situations. For example: Johnny and Su Lin were trying to figure out how much each of their puppies would weigh at ten months.

Johnny's Jack Russell								
Month	3	4	5	6	7	8	9	10
Weight	14	20	25	29				

Su Lin's Great Dane									
Month	2	3	4	5	6	7	8	9	10
Weight	10	18	26	32	38	42			

Ask students to work it out using the tables. Ask: How did you decide how heavy each dog would be? How did the weights change for each dog?

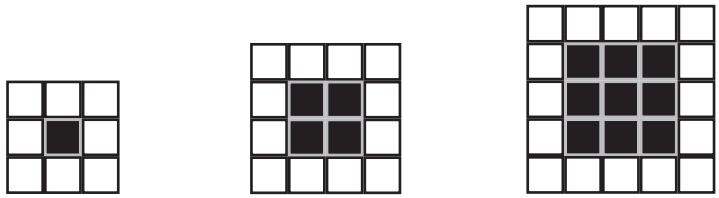
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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Picture Frames

Have students build a sequence of squares with a white border and black central tiles, and represent different variables in numbers.



Position in the sequence:	1	2	3	4
Length of the black square:	1	2	3	4
Number of black squares used:	1	4	9	16
Perimeter of black squares:	4	8	12	16
Number of border squares:	8	12	16	20
Perimeter of border:	12	16	20	24

Draw out that different aspects of the situation can be represented with numbers depending upon the question to be answered. Different aspects (variables) have different patterns.

River Crossing

Present the following problem: *A group of eight adults and two children want to cross a river. Their boat can hold just one adult or up to two children, but not an adult and a child together. Everyone can row the boat. What is the minimum number of one-way trips needed for all to cross the river?* Have students act out, draw diagrams or use other methods to solve the problem. Ask: How can you represent that situation with numbers? Why is this helpful? (See Case Study 2, page 241)

Triangle Toothpick Design

Have students copy this design with toothpicks:



Ask them to write number sequences to answer the following questions: How many triangles in each shape? What is the number of toothpicks needed to make each shape? How many diagonals in each shape? What question could have been asked to get a number pattern like 1, 1, 2, 2, 3, 3, ... from the design?

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Position in the Sequence:	1	2	3	4
Number of Triangles:	1	2	3	4
Number of Toothpicks:	3	5	7	9

Folding Paper

Have students represent situations that grow rapidly. For example: Record the number of regions created by successive folds of a sheet of paper. Record how the number of regions grows (after one fold there are two regions, after two folds there are four regions). Fold to see how many regions for four folds. Predict how many regions there will be for five and six folds.

Rumour Mill

Invite students to investigate different situations that can be represented with the same number sequence. After the Folding Paper activity above, present this scenario: A rumour started at school with one student telling two others that there was to be a new holiday on Sir John A. MacDonald's birthday January 11. The two students were told that next day they must each repeat the rumour to two more students. Each of these new students was to repeat the rumour the next day, to two more students. If the rumour started on January 2, how many students would have heard it by January 11? How do you know? How is this problem similar to the paper-folding problem? Draw out that representing the situations with numbers enables them to see that the same pattern exists for both.

Painted Cubes

Have students solve problems by representing them with a number sequence.

- The letters A to M are being painted on wooden cubes. A is painted on one cube, B is painted on two cubes, C is painted on three cubes and so on. How many blocks are painted altogether?
- Investigate the number of handshakes there would be in a group of people. How many for two people? How many for three people? And so on.
- Investigate joining dots on a circle. How many lines for two dots, three dots, four dots and so on?
- The T-ball committee is planning to have each of the six teams play each other once. How many games is that? What about for any number of teams?

Ask: What do you notice about the number sequences for the different situations? What is the pattern? Draw out that different situations can be represented by the same pattern.

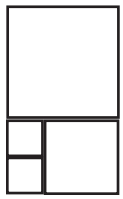






Fibonacci

Ask students to follow rules to produce the Fibonacci sequence shown right. Use 1-cm grid paper. Draw a 1-cm square near the centre of the page. Draw another square above that. See the two squares as a rectangle and draw a square onto the right of this rectangle. Draw a new square onto the long side of the rectangle. Keep drawing new squares onto the longest side of the rectangle.

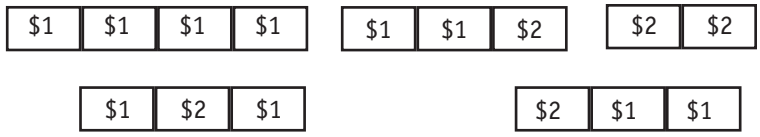


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Ask students to record the length of the sides of each new square as a number sequence. Ask: Can you see a pattern? Use the pattern to predict the lengths of the squares beyond those that you have already drawn. (The sequence will be 1, 1, 2, 3, 5, 8, ... which is the Fibonacci sequence.)

Money Patterns

Encourage students to look for common number sequences in the problems they solve. For example: For a garage sale, you have $1 and $2 stamps that you are planning to use to price various items. Draw diagrams to show the different arrangements of stamps to make up totals of $1, $2, $3, $4 and so on. (Reflections are included.) Draw and record the number of arrangements for each price. For example: There is only one way to make $1. That is a $1 stamp. For $2 you can have two arrangements: two $1 stamps or a $2 stamp. For $3 there are three possible arrangements: three $1; a $2 and a $1; or a $1 and a $2. For $4 you could have:



Ask: What kind of number sequence is this? Look out for other problems that have this sequence.

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CASE STUDY 2

Sample Learning Activity: Grades 5-8—River Crossing, page 238

Key Understanding 2: Representing aspects of a situation with numbers can make it easier to see patterns in the situation.

Working Towards: The end of the Factoring and Operating phases

BACKGROUND TO THE TASK

Mr. Bryson presented his class of eleven- and twelve-year-olds the problem at the right to solve. Some groaned when they saw the problem, commenting that it was "old hat". Many started to move toothpicks and counters across an imaginary river and others drew diagrams to represent the trips and crossings.

There was some discussion about how to minimize the number of trips. Christian said: *You have to send two children to start with, because if you only send one child or one adult they just have to come back to get another child, so it's a waste of a trip*. Eventually they all agreed that the minimum number of trips suggested by the majority of students (33 trips) was correct.

A group of eight adults and two children want to cross a river. Their boat can hold just one adult or up to two children, but not an adult and a child together. Everyone can row the boat. What is the minimum number of one-way trips needed for all to cross the river?

CHALLENGE TO THE CURRENT WAY OF THINKING

Most thought their task was complete and were surprised when Mr. Bryson asked: *What if there were different numbers of adults? What is the minimum number of trips for six adults and two children to cross, or 15 adults and two children, or 100 adults and two children?*

Initially many worked through the additional problems in the same way they had worked out the first problem. However, several began to anticipate the tediousness of working through the 100-adult example, and groaned in an exaggerated way.

Mr. Bryson used this as the excuse he needed to suggest that if they looked for some kind of pattern in the river crossings, they might find an easier way to solve the problem for 100 adults. The class had spent a fair bit of class time on "being systematic" when looking for patterns or winning strategies for games, so the students had some previous experience in recording the steps in a problem solution. They began to focus on recording their actions, rather than just the results of the actions. Many drew pictures or diagrams. Others used a tally.

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Tara and Michelle demonstrated a pattern they had found with counters. *First you send two children over, next you bring one child back, then you send an adult over, and the other child brings the boat back. It doesn't matter how many adults you have, you just need two children.* Andreas said: *Look, I can do it over and over the same way—two children cross, one child back; one adult cross, second child back; two children cross, one child back and so on.* He drew a line under the second child back to show where the pattern started again. Sarah commented: *Every four trips were the same—you just do it over and over and they all get across.*

Eventually, most students could see and explain the repetitive pattern in the minimum strategy for getting across the river. However, few were able to use this to work out the minimum number of trips. At best they could say things such as: *It helps because you know it's the same pattern over and over no matter how many adults there are;* or *You just have to keep track so you know when to stop repeating it.* Sarah's comment came closest, but even she was unable to see how to use her generalization to say how many trips it would take to move 100 adults.

Mr. Bryson was not surprised by this—even seeing how the crossings should be made does not make a general rule "pop out". This is a good example of a problem where the number pattern is easier to deal with than the actual situation, which was why he had chosen it.

So Mr. Bryson decided to intervene, saying briskly: *Why don't we get systematic and put the information we have in a table?* He quickly drew a table on the board and put in the data the class had, commenting that it often helps to organize the information in order. Some students had tried different numbers than those Mr. Bryson suggested and the class put those in. Mr. Bryson then suggested that the class try to fill in some gaps. *Where will we begin? With the easy numbers or the hard ones?* The class then assigned the numbers below six with several students doing each number (as a check) and filled in the information.

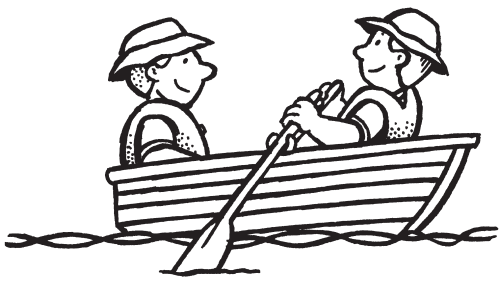
Number of Adults	Number of Trips Needed
6	25
8	33
12	49
15	61

Number of Adults	Number of Trips Needed
1	5
2	9
3	13
4	17
5	21
6	25
8	33
10	41
12	49
15	61

Immediately the pattern jumped out at them. *Every extra adult takes four more trips,* said one. *That's what I said before,* said Sarah.

Mr. Bryson asked: *How many for one adult?* They all said: *Five.*

How many for ten adults? Mr. Bryson asked. A couple of students started to blurt out, 50, but stopped almost immediately, laughing sheepishly. *Yes, how do we know it can't be 50?* Mr. Bryson asked. After a pause, looking at the table on the chalkboard, Tao said that it would be between 33 and 49: *It would be more than for eight adults and less than for 12 adults.* After another short pause, several students volunteered that it would be 41. One said it would be two more fours after eight adults.



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So how many for 100 adults? Mr. Bryson asked. Pause. *Don't all shout it out at once*, Mr. Bryson said laughingly, *talk it over with your partner and when you agree, write it down*. Within a few minutes, most of the students had decided it was 401. A couple initially said 410, but were dissuaded by others. A few began by counting forward on their calculator in fours. Most either began by jumping 40 for each additional ten adults, or followed the lead of those who had. However, there were a small number who gave an answer almost immediately, having multiplied 100 by four and added one.

At this stage, Mr. Bryson decided to draw out the Key Understanding that representing a situation with numbers can make it easier to see the patterns. He pointed out that they had all done an excellent job of working out the minimum strategy and could see the pattern in the trips themselves, and some had even noted that four was significant. However, turning it into an easy rule to decide how many trips for any number of adults proved quite hard. Letting them struggle with the diagrams for a while was part of Mr. Bryson's strategy for emphasizing how the pattern of the numbers helped. *In this case,* he said, *You knew how to do the trips in the minimum way and so had a strategy for doing it for 100 adults, but it was only when you looked for a pattern in the numbers that you were able to quickly say how many trips it would take.*

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Key Understanding 3

To describe a number pattern means to provide a precise rule that produces the pattern.

Within a mathematical context, to describe a number pattern means to provide an unambiguous rule or relationship that produces it. Students should be able to follow rules provided by others, create rules for themselves and produce rules that fit the information provided. (See Background Notes, page 210.) There are some conventional mathematical types of rules that students should begin to use in Kindergarten to grade 3. For example:

- Sequences of numbers can be described by giving a rule that says where to start and how to get from any number in the sequence to the next one. For example: Start with seven. Each number after that is five more than the one before (7, 12, 17, 22, ...).
- Sequences of numbers can also be described by giving a general rule that says how to work out any number in the sequence by knowing what its position in the sequence is. For example: Each number in the sequence is two added to five times its position (7, 12, 17, 22, ...).
- Other patterns can be described by rules that say what the general relationship is between two quantities. For example: The area of a square is the square of the length of its side.

From Kindergarten, students should be encouraged to use their everyday language to talk about the patterns they have observed, created or produced according to rules provided by others. In grades 3 to 5, they should learn to clarify and refine their descriptions, using as the criteria that another person should be able to recreate the sequence or arrangement from the pattern description alone. Thus, describing 6, 12, 24, ... as a "doubling pattern" is not enough, but saying "start with six and then keep doubling" is. In grades 5 to 8, students should also get better at writing rules for patterns. Trying to follow the rules of others will help them to identify what is needed in a rule.

Some ideas about describing patterns are provided in the Background Notes on page 210.

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Students should come to understand that sometimes there is enough information to make it possible to work out for sure what the pattern must be, even if we personally find it difficult or even impossible to work out. At other times, there is not enough information to be sure what the pattern must be. For example, we might only know the first three terms of a sequence, such as 1, 2, 4, ..., However, there may be enough information to work out what the pattern could be or is likely to be.

Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ use language such as "over and over," "repeat," and "again" to describe patterns and to say what is the same about two versions of the same pattern
Partitioning	■ produce number sequences by following rules based on addition and subtraction, such as "begin with 1, 2, and then each number is worked out by adding the previous two numbers" ■ can recognize ambiguities in rules given to them by others, and, with prompting, attempt to test their own rules to remove ambiguities ■ understand why they need both the starting number and constant multiplier in order to generate the sequence 4, 12, 36
Factoring	■ describe sequences within their repertoire sufficiently well to enable a peer to reproduce them, although their descriptions will generally be in natural language and involve describing the relationship between one term and the next
Operating	■ are able to describe the relationship between a term and its position in the sequence directly ■ write simple rules that describe the relationship between two quantities ■ rewrite rules expressed in natural language in shortened form to make them easier to follow

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Number Scrolls

Invite students to skip count with a calculator, record the numbers on cash register tape, and say what number will come next and what number they might reach at the end of the tape. Ask: How did you decide? Extend the activity by having students use a rule to create and write a number pattern, then leave out part of the sequence and have a partner find the rule to fill in the missing part.

What's My Rule?

Ask students to predict missing parts of a sequence by first working out the pattern, then saying what the missing part must be. For example: A student makes a pattern of four repeating units using Pattern Blocks behind a barrier and covers two of the units. The student then asks a partner to look at what is not covered and make the hidden part. Extend the activity by using different objects for the same number pattern.

What's Next?

Extend *What's My Rule?*, above, by having students predict what the next object will be in their partner's sequence. Ask: Why would that be the right piece? What will the tenth piece be in the sequence? What about the 20th?

Recording

Use the activities *Necklaces*, *Lining Up*, and *Stairs* on page 234, for students to record all the numbers in a sequence and say what the rule is that generated the pattern. For example: *It's two one, repeated over and over.*

Story Patterns

Have students count objects illustrated in stories to see if there is a pattern, and predict how many will be on the next page. For example:

- *Ten in the Bed* (Dale, 1999). During the reading, ask: How many things do you think will be left in the bed next? Why do you think that?
- *The Hungry Caterpillar* (Carle, 2000). Ask: How many things will he eat the next day? Why do you think that?

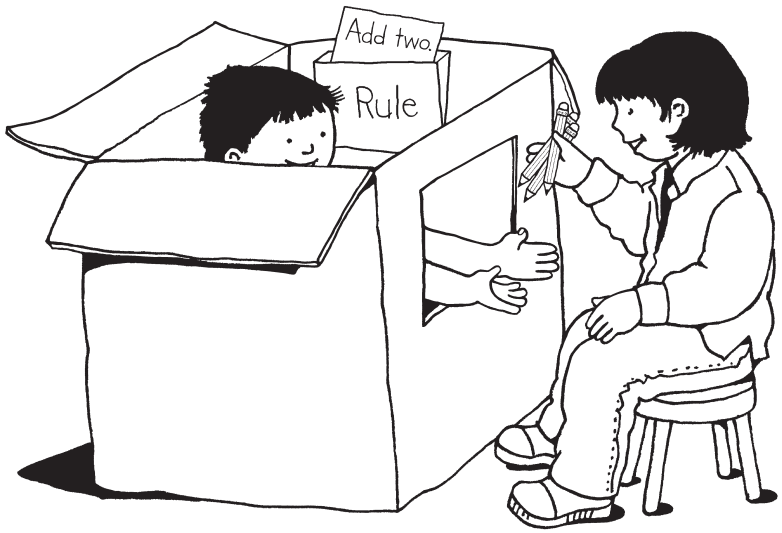
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How Many?

Have students use a number pattern to plan how many objects they will need. Ask them to choose a number pattern from the class chart made in the Reference Chart activity on page 235, say 1, 4, 2, and choose from three containers where there are twice as many shells as buttons and beads to make a pattern. Help them plan how many of each of the objects they will need if they make three repeats.

Function Box 1

Invite students to work out a rule that is used to change one number of things into another number of things. For example: A student sits inside a box with a collection of pencils and a rule such as "add two". Another student feeds in three pencils through a slot. The student inside adds two, wobbles the box and feeds out five pencils. Others look at the change to the number and say what the function of the box is.



Function Box 2

Repeat *Function Box*, above, using a note with a number on it, such as five. The student inside the box crosses out the five and replaces it with seven, and feeds the note out. Repeat with different numbers for others to say what the rule is. Include doubling, halving and subtraction.

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

Objects

Have students create a pattern along a length of card, using their choice of materials. Ask them to record the pattern by writing the number of objects for each part of the pattern below the materials. They then remove the objects and give the card to a partner to recreate the pattern using different objects. Ask: How is the second pattern similar or different from the original? How can you ensure that your partner recreates exactly the same pattern? Have students write a rule in words to describe their original pattern and pass this to their partner. Ask: Does this help to create the same pattern? Why?

Bicycles and Skateboards

Following *Tricycles*, on page 236, have students generate similar number sequences for the parts needed to make different numbers of bicycles and skateboards. They then write a rule to help a new factory worker decide how many of each part will be needed for any number of bicycles/skateboards. For example: For 64 skateboards you would need to multiply the number of skateboards by four to find the number of wheels.

Number of Skateboards:	1	2	3	4
Number of Boards:	1	2	3	4
Number of Trucks:	2	4	6	8
Number of Wheels:	4	8	12	16

Calculator Patterns 1

Have students create a number pattern on a calculator, then write a rule for the pattern. For example: Enter 1 2 + 3 then press = = = to create 12, 15, 18, 21, ... They pass their written rule to another student to read and create the number sequence described. Ask: How are the sequences the same or different? What part of the description needs to be changed to make sure everyone writes the same sequence? Extend the activity by asking students to predict whether a given number will appear in their sequence as they use the constant function. For example: 1 4 + 8 = = = ... 22, 30, 38, 46, 54, 62, 70 ... Ask: Will 113 be in the sequence? How do you know?

Function Box

Extend *Function Box*, on page 247. Have students create a collection of rules for a student in the function box to apply to the numbers pushed through the slot. Then the students who wrote the rule say whether the answer is the one they expected. Ask: If not, why not? Suggest to the students that they refine and rewrite the rule so that it always gives the expected answer.

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Ask: What number sentences could be used? In what order should the numbers in the sentence be? Does it matter? Why?

Same Answer

Have students place the rules created for the previous *Function Box* activity into groups that give the same answer. Ask: Why is it that "add four take away two" gives the same result as "take away three add five"?

What's My Rule?

Ask one student to think of a rule for others to guess, such as "add four" or "times two". Have students give a number and the leader uses the rule to give the answer. After five turns of calling out a number to the leader, students decide what rule is used. Compare the suggested rules with the original. Ask: How are they the same (different)?

What Comes Next?

Have students write the next three numbers in sequences and write their rule for each. For example:

- 2, 9, 16, 23, ☐, ☐, ☐
- 16, 15, 13, 10, ☐, ☐, ☐
- 1, 2, 4, 7, 11, ☐, ☐, ☐
- 3, 3, 6, 18, ☐, ☐, ☐

List the different number sequences the students make for each example. Ask: What rule did you use to write the numbers? Why is it possible to think of different rules for each sequence?

Roman Numerals

Have students write numbers 1 to 20 using Roman numerals and decide on the patterns and rules used as its basis. Ask them to test the rules by writing bigger numbers, such as 328, 1054, 1998, 50 348. Ask: Does your rule work for these? If not, how can it be modified so that it does? Have students investigate the patterns and rules in other number systems and compare to our Hindu-Arabic system.

Seven

Ask students to investigate how many times the number seven is used in a book with 87 pages. Ask: How many fives in the same book? How do you know? Have students write a rule to find the number of pages for a book of any size, then test the rule.

Homework Patterns

Have students use different rules to describe and extend a number sequence. Present this scenario: Sam and Maya were doing their homework. They had to look at the pattern and fill in the box. The pattern was 50, 62, 74, ☐, 98. They both put 86 into the box. The next instruction was to continue the sequence from 98. Maya wrote 110, 122, 134. Sam wrote 1010, 1112, 1214. Ask: What rule do you think Sam used? What rule do you think Maya used? Who do you think was correct? Why?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Triangle Toothpick Design

Have students find a rule to describe a pattern. Ask them to make the first four shapes of this design.



Write a rule to say how the number of toothpicks changes with each new shape. Ask students to write a rule in their own words, which connects the position number of the shape to the number of toothpicks. Have students exchange rules with their partner and use their rule to find the number of toothpicks in the next few shapes. Ask: Did the rule work? Use the toothpick pattern to explain why their rule works. Ask: Would your partner be able to predict the number of toothpicks for any shape position using your rule? Why?

Different Rule, Same Pattern

Extend *Triangle Toothpick Design*, above, so students see that apparently different rules can produce the same pattern. Present this scenario: When Lee and Allison swapped rules for their triangle toothpick designs, they discovered that their rules sounded different. Lee said: *You can say it's three, add the number of the shape less one, times by two.* Allison said: *You times the number of the shape by two and then add one.* Ask: Has someone made an error or do both rules work for this pattern? Have students use their toothpicks or diagrams to explain their reasoning.

Hexagon Patterns

Extend *Different Rule, Same Pattern*, above, to another design. Present this scenario: Students were making a hexagon toothpick pattern.



Jeremy and Sophie swapped rules and discovered that their rules sounded different. Jeremy said that the number of toothpicks was equal to five times the position number of the shape, add one, and he wrote it as $5 \times n + 1$. Sophie said that the number of toothpicks was six times the number of the shape, take away the number of the shape less one. She wrote it as $6 \times n - (n - 1)$. Ask: Do both rules work for this pattern? Use the toothpicks or diagrams to show why each rule works. Find some other rules that work.

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Sum Patterns 1

Invite students to find rules to make calculating easier. Ask them to examine these number sentences: 1 + 2 + 3 = 6, 2 + 3 + 4 = 9, 3 + 4 + 5 = 12, 4 + 5 + 6 = 15 and say what patterns they find. Ask: What are the next three number sentences? Repeat the activity with these number sentences: 1 + 2 + 3 + 4 + 5 = 15, 2 + 3 + 4 + 5 + 6 = 20, 3 + 4 + 5 + 6 + 7 = 25, 4 + 5 + 6 + 7 + 8 = 30. Ask students to write more for a partner to find the answer without adding the numbers.

Sum Patterns 2

Extend the activity above with this scenario: Gemma came up with the rule that you can multiply the middle number by three to get the answer. Tom said that with four consecutive numbers you multiply the second number by four to get the answer. Ask: Do these rules work? How do you know? Find rules that do work for adding four or five consecutive numbers.

Everyday Formulas

Have students recognize formulas used in everyday life, such as money conversions. Present this scenario: Alex and his family visited a museum in New York. The cost of admission was listed in Canadian and American dollars: adults $15.00 US or $18.25 CAD, students $8.00 US or $10.00 CAD, Seniors $10.00 US or $12.50 CAD, and Families $25.00 US or $31.25 CAD. Have students organize this information to help Alex work out an exchange rate for Canadian to American dollars.

Magic Calculating Machine

Ask students to describe relationships between two quantities. For example: Display a diagram of a magic calculating machine. Tell students the machine will apply a certain rule to any input number to create the output number. For example: when seven is put in, 3.5 comes out; when 4.5 goes in, 2.25 comes out; three in, 1.5 out; ten in, five out. Ask: What rule is used to calculate the output numbers? How can you be sure?

Graphs

Have students find and compare rules that connect points on a graph. For example: Plot these points on a graph—(14, 16), (0, 2), (5, 7) and (12, 14). Ask: How are the pairs of numbers linked? What could the rule be? Ask students to use their rule to plot the other numbers up to ten (horizontal axis). Ask: How do you know your rule works? Now graph this rule: Double the number on the horizontal axis, add four and divide by two. What do you notice when you compare the two graphs? How can the graph have two different rules?

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CASE STUDY 3

Sample Learning Activity: Grades 5-8—River Crossing, page 238

Key Understanding 3: To describe a number pattern means to provide a precise rule that produces the pattern.

Working Towards: The end of the Factoring and Operating phases

BUILD ON EXISTING KNOWLEDGE

Mr. Bryson developed this lesson to extend the *River Crossing* activity in Case Study 2 on page 241. In particular, Mr. Bryson wanted the students to develop their capacity to express rules in general terms that can be applied to any suitable number.

Almost all the students had decided upon 401 trips for 100 adults. Mr. Bryson decided to return to the couple who seemed a little unsure later and to press on. Mr. Bryson did so by offering another challenge: *I know I started out by saying I only wanted to know how many trips were needed for 100 adults, but now I am going to really challenge you. How many for 1000 adults and the same two children?*

Almost immediately their hands went up. The general consensus was 4001, which Mr. Bryson entered in the table. Mr. Bryson then asked Andrew if he could explain how he did it. *It was four groups of 1000 and one more*, Andrew said confidently. *How did others do it?* Mr. Bryson asked. Several students offered their approaches, which were essentially the same. *So what if there were 25 adults?* Mr. Bryson asked. *104!* the students said.

At this point, Mr. Bryson asked students to work with a partner to write an instruction that would help someone who had not been through this with the class to work it out for any number of adults. He then asked volunteers to record their rules on the board. Most were variations of the following:

- how many adults times by four and add one
- times four by the number of adults add one

Mr. Bryson then asked students to test each rule on 25, 100 and 1000. Realizing that they would probably apply the rule meant rather than what had actually been written, he also "tested" them, and for the second rule above wrote:

25 adults —> 104 100 adults —> 404 1000 adults —> 4004

The class then compared the expressions and agreed that although some looked different they were all mathematically the same, because they all produced the same pattern of trips. Some had forgotten about the extra trip needed at the end for the two students to complete the crossing, but quickly realized where the one in the rule came from when peers pointed it out.

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The students immediately pointed out that Mr. Bryson was wrong. *But that is what the rule says*, Mr. Bryson said, *four times by* (pause) *the number of adults plus one, so that's four times* (pause) *25 add one, which is four times 26, which is 104*. The students hastened to explain that Mr. Bryson had to multiply first and then add one. *But it doesn't say that*, he complained. *How can we make it clear?* After a few moments, the class had an alternative:

- times four by the number of adults and then add one

Mr. Bryson suggested that students revise their rule if they needed to and give it to another pair of students to test. *Make sure you are testing what people really wrote*, he said, *and not what you think they meant to write*. After, students were happy that their rules were clear and fitted all the number pairs in the table, and the class refined any rules on the board that needed it.

Mr. Bryson then asked if they could shorten their rules, prompting them by writing:

- The number of trips needed =

Students then produced a series of more simplified rules:

- The number of trips needed = how many adults x 4 + 1
- The number of trips needed = (4 x the adults) + 1
- The number of trips needed = the number of adults x 4 + 1
- The number of trips needed = 4 x number + 1

Although Mr. Bryson knew that we would later want students to be able to reduce this further to the algebraic expression 4 x n + 1 or 4n + 1, he did not pursue it. Instead, he focused on the need to be sure that the rules fitted all the given data and were stated clearly. The class then compared the rules and decided that they all said the same thing.

LINKING BACK

At this point, Mr. Bryson felt that the key points about checking that rules fitted the data had come out. He then asked students to see if they could explain to each other why their rule worked. Again, Mr. Bryson prompted them to look back to their original problem.

Mr. Bryson was happy to find that most students realized that the multiplication by four in their rules related to the "counting by fours" pattern they had found when they added on the number of trips taken as each adult crossed the river. Mr. Bryson asked them to write in words what they had found and obtained a range of expressions that they decided all matched the "four times" number pattern.

Mr. Bryson allowed students to express their "rules" in personal ways and tried not to lead them to one "correct" rule. He felt that they needed to see that all the different number sentences they devised produced identical number patterns, and were therefore mathematically equivalent.

Number of Adults	Number of Trips Needed
1	5
2	9
3	13
4	17
5	21
6	25
8	33
10	41
12	49
15	61
25	101
100	401
1000	4001

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Key Understanding 4

There are strategies that help us become better at recognizing common types of patterns.

This Key Understanding should develop in conjunction with Key Understandings 2 and 3. Students should come to see that pattern recognition is more than "look and see" or luck or ability. They will become better at identifying patterns by using good pattern-finding strategies. For advice on this, see Background Notes, pages 212 to 217.

Students should investigate a range of pattern types, but simply providing variety without structure is unlikely to be helpful. They are more likely to recognize pattern types with which they have had systematic experience and where their attention is focused upon:

- *The similarities between certain patterns*: They should look for what is the same and what is different between various patterns, leading to simple classifications of patterns.
- *The strategies that they found helpful in identifying patterns*: It can be difficult to recognize a pattern immediately, "just by looking"; having some "search strategies" is essential.

The repertoire of numbers and operations involved in pattern-searching activities should expand throughout Kindergarten to grade 3, as should the complexity of the patterns.

Links to the Phases

Phase	Students who are through this phase. . .
Matching	■ can recognize counting and repeating patterns based on whole numbers
Quantifying	■ has expanded to include sequences involving addition and subtraction of a whole number and patterns in addition tables and a 100-chart
Partitioning	■ use strategies to identify patterns involving each of the operations on whole numbers ■ can describe how terms in a sequence are linked by multiplication or a more complex addition/subtraction-based rule
Factoring	■ apply their pattern-finding strategies to sequences involving any one of the four operations on whole numbers, decimals, and fractions and to relating pairs of numbers in simple "guess my rule"-type games
Operating	■ can identify patterns in sequences or sets of number pairs that are based on more than one operation

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Stairs

When making a 1, 2, 3, 4, 5, 6 pattern, such as building a staircase, have students say what the change in the numbers is each time a new step is added. Ask: How many squares will be in the next step? Why do you think that? Check to see.

Identifying Patterns

Have students identify and record number patterns, such as dots arranged in growing patterns on the overhead projector:



Ask: What is the difference between one and two, two and three? What is happening each time? Could the next number be seven? Why? Ask: How is this pattern the same as the "stair" pattern?

Search Strategy

Encourage students to develop a search strategy that focuses on the difference between the first and second terms, second and third terms, and so on. For example: In the 1, 2, 3, 4, 5, 6 pattern, help students to see the constant difference is "add one".

Find the Rule

Use *Search Strategy*, above, to find the rule for a given sequence of numbers, such as a previously generated number scroll: 2, 4, 6, 8, 10, 12. Have students find the numbers on a number line and work out the difference between each of the numbers, then test out their conjecture using the constant function on the calculator to generate the sequence.

Practising Strategies

Have students practise using pattern-searching strategies and make a chart for each strategy. Ask students to record the number sequences they were able to work out using that strategy. For example: By comparing adjoining numbers in the sequence 2, 4, 6, 8, students see that a pattern is generated by repeatedly adding two. They may name the strategy as "Write What Changes From One Number to the Next".

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K-Grade 3: ★ Introduction, Consolidation or Extension

Make a Pattern

Use a beginning unit to generate a pattern, such as one acorn, two leaves, one acorn. Students may come up with different rules. They may see the unit as 1, 2 and repeat that, or they may repeat the 1, 2, 1 unit. Others may add one to each new unit: 1, 2, 1, 2, 3, 2, 3, 4, 3. Use this beginning unit several times for students to focus on using different rules to make different patterns.

Fairy Tale Steps

Use familiar fairy tales, such as Jack and the Beanstalk, to have students use a rule to generate different number patterns. For example: Draw a stepping stone number line for students to say what their rule is and then move the fairy tale characters along—the giant uses an "add five" pattern, Jack uses "add two". Ask students to record the number sequence made by each rule.

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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Farmyard

Ask students to use ten blocks to create a farmyard for some plastic horses, cows or chickens. Ask: How many blocks will be needed for a farmyard one layer high, two layers high, three layers high? Have students create a number sequence to describe the sequence of layers. Change the fence so that it has 12 blocks for the first layer and then write a number sequence to show how many blocks are needed for two layers, three layers, and so on. Ask: How is this number sequence the same as the last sequence? How is it different?

Constant Function

Have students use the constant function on a calculator to generate other doubling patterns, such as 3, 6, 12, 24, ... Ask them to compare their patterns with others and say how they used the calculator to generate the sequence. Ask: How does the size of the numbers help you to know if a sequence you are given is a doubling pattern? What is the difference between the consecutive numbers in the sequence?

Halving

Invite students to investigate halving sequences using cash register tape strips. Ask them to choose a strip between 30 and 40 cm long. They fold the strip, measure and record the length of half, fold it in half and record, fold again and record. Have students compare sequences. Ask: What is happening to the numbers in each? How could you generate the same sequence using a calculator? What is the difference between the consecutive numbers in the sequence? How is this different from a subtraction sequence?

Three

Have students choose their own starting number and generate a sequence using the constant function on a calculator to + 3 each time. Start with the same number and generate another sequence using – 3, then another using x 3 and another for ÷ 3. Ask: What is the difference in the sequences? Which one increases quickly? Which increases slowly? Which decreases? Why? Does this happen for all +, –, x and ÷ sequences? Try starting with 24 and using x 0.5 to generate a sequence. Ask: Why did this sequence get smaller?

Create a Sequence

Following *Three*, above, have students start with the same number, such as 12, choose one sign and use this with a set number, such as 4, to generate their own sequence. Ask them to give their partner the sequence and see if they can say which sign was used and how they know.

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Grades 3-5: ★ ★ ★ Major Focus

Doubling

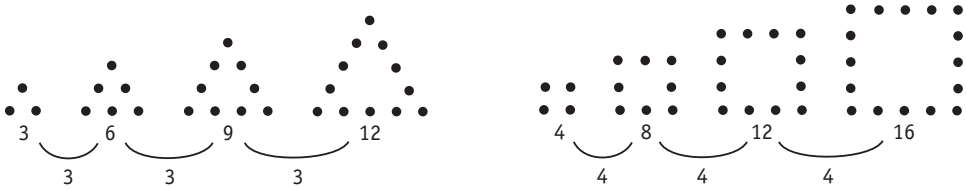
Have students investigate a doubling pattern by folding a rectangular sheet of paper. One fold gives 2 rectangles, two folds give 4 rectangles, three folds give 8 rectangles. Have students tabulate the results. Ask: What happens to the number of rectangles as the number of folds increases? From this, predict the number of rectangles for five, six and seven folds. Ask: Is there a limit to the number of folds that can be made?

Triangular Numbers

Ask students to investigate the sequence of triangular numbers after creating them in various ways (See *Triangular Numbers*, page 237). Ask: What is the difference between one term and the next in the sequence? How can you use this information to say if the following is a sequence of triangular numbers: 45, 55, 66, 78?

Dot Patterns

Have students create number sequences for a series of dot patterns and then say how the sequences are the same. For example:



Ask: What is the difference between the numbers in each sequence? How is the difference related to the shape? What would be the difference between the terms in the sequence for hexagons? (Note: These are not the triangular numbers, nor the square numbers.)

Function Box

During *Function Box*, on page 248, have students work out what the rule is by writing a list of the "in" and "out" numbers and then organize this into a table. After finding a few rules in this way, students compare the tables of numbers and look for similarities in the differences between the "in" and "out" numbers. Ask: How will we know if the person has used addition or subtraction in their rule?

Puppies

Extend *Puppies*, on page 237, so that students work out the changes in the dogs' weight each month and use this to predict their weight at ten months. Ask: Was the difference in the dogs' weight the same for each? How are the two patterns the same? How are these patterns different from other addition patterns, such as the dot patterns above?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ Major Focus

Pattern Pairs

Have students identify patterns linking pairs of numbers in grids and tables. For example: You are making homemade popsicles and need to make up a price list. Fill in the missing prices. Ask: How does the price change? Can you think of a rule to help work out how much any number of popsicles would cost?

Number of popsicles	1	2	3	4	5	6	7
Cost			45c	60c			

Extending Patterns

Ask students to find the difference between terms in order to extend a pattern, such as the *River Crossing* pattern in Case Study 2 on page 241. Ask students to use a table to record the trips needed for one adult, two adults and three adults. Ask: Can you see a pattern? What do you think the next number in the sequence will be? How do you know?

Times Tables

Have students find patterns in the 13 times table in order to continue the table without a calculator. Ask: What strategies did you use to extend the pattern? Repeat this with other difficult times tables.

Sticky Instructions

Ask students to write instructions for the continuation of number sequences. Write a range of number sequences on strips of paper. For example:

24, 21, 18, 15	2, 3, 5, 8, 12
1, 4, 7, 10, 13	10, 15, 19, 22
1, 4, 9, 16, 25	1, 1, 2, 3, 5, 8
6.4, 3.2, 1.6, 0.8	1, 3, 7, 15, 31
1, 3, 6, 10, 15	$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}$

Ask students to attach sticky labels to the strips with instructions for continuing the sequence. Have students use others' instructions to continue the sequences.

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Grades 5-8: ★ ★ ★ Major Focus

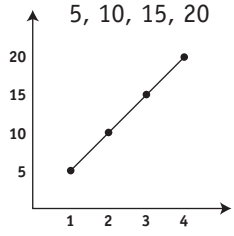
Classifying

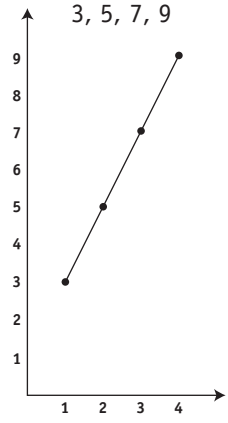
Have students examine and classify number sequences, such as those in the *Sticky Instructions* activity on page 259, according to the type of differences between the terms. Ask: Are the differences all the same? Where the differences are not the same, students can find out what happens when you find the difference of the differences. How could this group be re-sorted?

Graphs

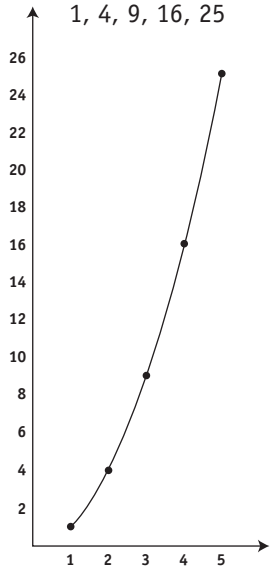
Extend the *Classifying* activity above by having students draw graphs of a range of number sequences. Ask: Are there graphs that seem to have a similar appearance? Have students sort and classify the graphs according to how they seem to grow.

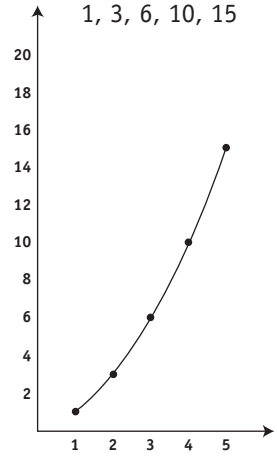
Constant addition





Adding an increasing amount





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Problem Situations

Have students classify common number patterns that represent problem situations. Over time, have students record and display problems, along with the number patterns used to solve them. Ask: What kind of problems have sequences that go up by a constant amount? Which problems have number sequences that grow rapidly? Are there other different kinds of number sequences?

Sum Patterns

Revisit *Sum Patterns*, on page 251. Ask: How did you organize the information to make pattern-finding easier? What strategies did you use?

Relationships

Ask students to work out the relationship between terms in a pattern. For example: Ask students to experiment with a calculator to work out the rule used to produce the following patterns:

60, 600, 6000, 60 000, ... 60, 6, 0.6, 0.06, ...
60, 70, 80, 90, ... 60, 50, 40, 30, ...

Ask students to sort the patterns according to those in which they are constantly adding/subtracting and those in which they are constantly multiplying/dividing.

Triangle Toothpick Design

Have students organize the number of toothpicks needed for the first four shapes in a table. Ask: How does the number of toothpicks change each time? Does multiplying each term by two work as a rule? Do you need to add or subtract as well, to make it work? (One rule may be to multiply by two and add one.) Explain why your rule works using the toothpick design. What will the 20th term be? (For explanation of "term", see Background Notes, page 212.)



Picture Frames

Have students use tiles to represent the first four frames of the *Picture Frames* activity on page 238 and then find patterns that enable them to answer questions, such as: How many border squares are in the 20th frame? How many squares are there in the tenth frame? Have students find rules so they could answer the different questions for any frame number. Ask: What was it about the number sequences that helped you decide on the rule?

Pascal's Triangle

Invite students to research the famous mathematician Blaise Pascal to find out what Pascal's Triangle is. Present them with a diagram of the first six rows of the triangle. Ask them to continue the different number sequences in the triangle, say what the pattern is and what they did to find out. Ask: What did you do to continue the sequence 1, 4, 10, 20?

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

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Key Understanding 5

Our numeration system has a lot of specially built-in patterns that make working with numbers easier.

Patterns are the basic building blocks for students' understanding of our numeration system—the way we write and say numbers—and consequently, for their capacity to count with large and small numbers, understand the order and relative magnitude of numbers, and calculate. For example, it is recognition of the patterns in the way we say and write numbers that enables students to count backwards and forwards from any number in tens—an essential skill in mental arithmetic. Also, it is the patterns in the way we write numbers that enable us to know immediately what the result is of multiplying, or dividing, by ten, and that 187 cannot be a multiple of five. As such, numeration patterns are a major focus of many of the activities in the Number sub-strands Whole and Decimal Numbers, Fractions and Computations, and students should spend considerable time observing, describing, explaining and using patterns on number lines, 100-charts, and addition and multiplication tables.

However, the essence of this particular Key Understanding is on students:

- working out *why* numeration patterns occur
- realizing how such patterns can help them do other things

Firstly, students are more likely to develop an understanding of our numeration system if they try to explain why certain patterns occur. For example, if we shade the numbers on a 100-chart by starting with any number and repeatedly adding 11, we get a diagonal. Students may notice and even find this "pattern" interesting but think of it simply as a surprise, a piece of magic or a party trick, and learn nothing from it about how place value works. It is when they attempt to explain to themselves and others *why* we must get a diagonal, that such an activity has the potential to extend their understanding and use of place value.

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Secondly, students are more likely to appreciate the significance of patterns in number, and to be prepared to look for them in future, if there is some real and immediate "pay-off" for them in terms of things they can do more easily. For example, using patterns in the multiplication table for nines can make it easier to remember multiplication facts and also make mental arithmetic with nines easier.

Thus, while this Key Understanding will largely be developed through activities that are also used elsewhere, the suggestions here complement and extend those activities by making explicit the need for active reflection by students on the nature and role of pattern. As students progress through the elementary school mathematics program, they should show an increasing:

- capacity to explain (rather than simply notice or describe) the numeration patterns they observe
- appreciation of how seeing these patterns can help to make working with numbers both easier and more interesting

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Sample Learning Activities

K-Grade 3: ★ ★ Important Focus

Teens

Have students chant and sing counting rhymes involving one to nine. Extend to the teens. Record the numbers in order and draw out that the digits zero to nine are repeated in the same order (except that we do not write the first zero).

Decades

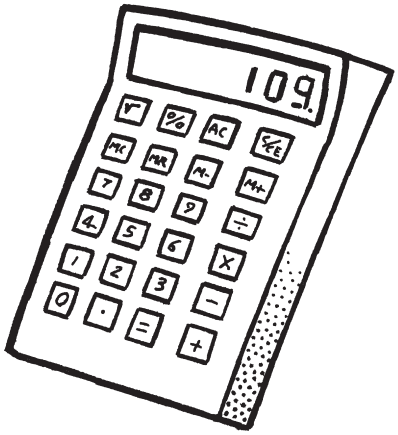
Record the counting sequence as students count aloud up, over and through the decades. Draw out that the digits (zero to nine) repeat within each decade and that the decades (tens) also repeat the digits in order (except that we do not write the first zero).

Number Sequence

Have ten sets of ten different coloured squares for students to make a number sequence from one to 100 to display around the room. They write one number per square, keeping the sequence of colours the same for every set of ten. Display the first ten and ask questions that encourage students to work out which number comes next, which colour comes next, which numbers used that colour before.

Hundreds

Using a calculator as a counting machine, have students count together and record the sequence from 95 (enter 9 5 + 1 = = = = and so on). Stop at 109 and ask: What will the next number be? How do you know? Continue counting and stop at 119. Ask: What will the next number be? Why? Continue up to and over 200.



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Constantly Adding Tens

Have students record the numbers from the calculator display on their own 100-chart (See Appendix: Line Master 20). Ask students to shade a specified number between one and ten, say six. They then add ten and shade again, and repeat until the numbers "run out" at 96. Students read the numbers 6, 16, 26, 36, ... and say what they notice about them. Ask students to then choose another number, say five, and predict which numbers will be in this list if they constantly add ten and shade the numbers. Ask: Why does every number in the list end in five? Why does it make a straight line up and down on this chart? Have students count the squares between the shaded squares to find the pattern.

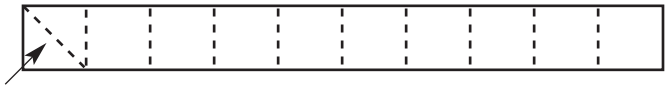
Make the Number

Have students use bundling material, such as Base Ten Blocks or straws bundled into ones, tens and hundreds, to show what is happening each time ten is added to, say 99, 109, 199. Ask: How does the material and the way the number is written match?

Number Scrolls

Invite students to generate number sequences by using the constant function on a calculator and record the sequences on cash register tape. Have students fold strips of cash register tape into equal-sized squares as shown below and record one number per square. Then ask students to generate lists of numbers by:

- repeatedly adding one. Ask: Why do the numbers after nine have two digits? Why do the numbers between 19 and 29 start with a two? What do the numbers after 29 start with? How many start with three? Why?
- skip counting in twos. Ask: What do you notice about the list of numbers in the ones column? What is the highest number in this column? Why? Look at 20. How many times did the calculator add two before it changed to 30? What about between 30 and 40? Is it the same between 50 and 60, 90 and 100? Why?



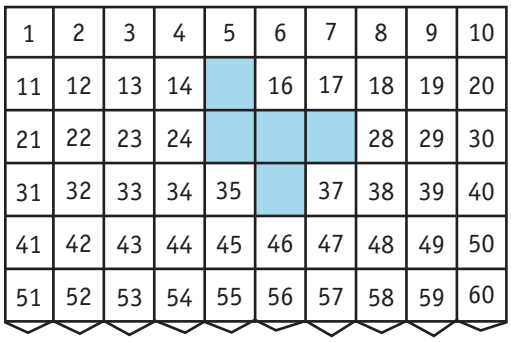
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Sample Learning Activities

Grades 3-5: ★ ★ ★ Major Focus

Page Numbers

Show students a book with at least 100 pages. Ask: How many times do you think seven is used in the page numbers? Why do you believe it will be that many? Check to see. Ask: Have we missed any? Are you sure? Can we explain why it is that number? Provide a 100-chart (See Appendix: Line Master 20) and encourage students to use it to explain why there are that many sevens. Ask: How many sixes will there be? Fives? Zeroes (the tricky one!)?



1	2	3	4	5	6	7	8	9	10
11	12	13	14		16	17	18	19	20
21	22	23	24				28	29	30
31	32	33	34	35		37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

Hide the Numbers

Provide various pentominoes—each made of five squares of the same size as those on the 100-chart (See Appendix: Line Master 21). Have students work in pairs and use a pentomino to cover parts of a 100-chart (See Appendix: Line Master 20) and say what numbers are hidden, explaining to their partner why they think that.

Move to other parts of the chart and repeat. Turn the pentomino and repeat. Draw students together and ask: How did you decide? What patterns did you use? How did the way our numbers are written help?

Constantly Adding Tens

Repeat *Constantly Adding Tens*, from page 265. Ask students to explain to their partner why when they begin with a number and constantly add ten, the resulting numbers are all in a column. Discuss as a whole class. Choose a larger number, such as 57. Ask: What happens if you constantly subtract tens? Why?

Constantly Adding Nines

Have students choose a number between one and nine, and shade squares on a 100-chart (See Appendix: Line Master 20) as they constantly add nine (without a calculator). They then compare their result with others. Ask: What is the same? What is different? Some students may correct their shading at this stage. If so, draw out that the pattern enabled them to notice their mistake. Ask students to work with their partner or group to explain why they always get that same arrangement of shaded squares when they add nines. Ask: Why does it "go down and back one" each time? Does it link to what happens when constantly adding ten? Link to nine being one less than ten.

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More Nines

Have students begin with any number less than 50 and shade squares on a 100-chart (See Appendix: Line Master 20) as they constantly add nine. Ask: Can you predict the shaded squares before you actually add the nine? What do you expect to happen if you subtract nines? Test. Help students explain why it happens, relating to nine being 10 – 1. After drawing out the pattern in adding nines, challenge students' mental arithmetic skills by asking: What is 37 + 9? (Down to 47 and back to 46.) Do several of these immediately and reinforce over following days and weeks.

Adding and Subtracting

Extend *More Nines*, above, to adding and/or subtracting any number between one and nine. For example: start at 34 and add seven repeatedly. Ask: Does thinking of seven as 10 – 3 help explain the arrangement of the squares? Does it help with mental arithmetic? (86 + 7 is three less than 96.)

Answer Patterns

Have students investigate the answer patterns in related sets of additions or subtractions, such as 15 – 8, 25 – 8, 35 – 8, 45 – 8. Ask them to explain why it happens, and use this to predict related calculations, such as 95 – 8.

100 Grid

Have students make a rectangular grid, arranging the numbers one to 100 in whatever number of rows and columns they wish. For example: Ask them to use their grid to look for patterns in the numbers. Then ask them to quickly find, say, 67 or 42. Place a 10 x 10 grid for all to see and ask: What changes from one row to the next? Why? What changes in the other grids? Why? In which grids is it easiest to find particular numbers? Why? Have all students make a 10 x 10 grid for their personal use.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18

Division Patterns

Encourage students to investigate division patterns. For example:

1 ÷ 2	10 ÷ 2	100 ÷ 2
2 ÷ 2	20 ÷ 2	200 ÷ 2
3 ÷ 2	30 ÷ 2	300 ÷ 2 through to 900 ÷ 2

Ask: Why do the answers have either a five or a zero at the end? How does the change in the size of the numbers in the questions affect the size of the numbers in the answers?

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Sample Learning Activities

Grades 5-8: ★ ★ ★ **Major Focus**

Hide the Numbers

Vary *Hide the Numbers*, on page 246 by using various hexominoes—each made of six squares of the same size as those on a 1000-chart constructed in rows of ten (See Appendix: Line Master 22 and 23).

Multiples of Nine

Ask students to shade in multiples of nine on a 100-chart (See Appendix: Line Master 20). Ask them to investigate patterns in the resulting display. Prompt where appropriate: Is there a pattern in the ones place of the multiples of nine? What do you notice? What is happening in the tens place?

Finger Patterns

Have students hold up their ten fingers and drop the first finger on the left. How many fingers can you see? (nine) Lift that finger and drop the second finger from the left. How many fingers to the left of the dropped finger are there? (one) How many to the right? (eight) So the second dropped finger makes one, eight. Drop the third finger (makes two, seven). Ask: Can you link what you see with your fingers with nine times table?



Ask: What do you notice? Why does this happen? How can it help you remember the nine times table?

Number Race

Display ten different three- or four-digit numbers to the class, such as 357, 985. Allocate the numbers 2, 3, 4, 5, 6, 7, 8 to different groups of students in the classroom. Have a race to see which group can be first to work out if their number is a factor of each of the numbers displayed. Ask: Was that a fair race? Why? What is it that makes that an easy question for some groups and not others?

Working to Rule

Have students use the 100-chart (See Appendix: Line Master 20) to explain why rules work. Ask students to draw a 3 x 3 square anywhere on the 100-chart, and to add the corner numbers of their small square. For example: The corner numbers might be 38, 40, 58 and 60 and you get 196 when you add them. Divide the total by four. What do you get? Repeat this process with

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other 3 x 3 squares. What do you notice happens each time? What rule can you use to easily calculate the sum of the corner numbers? Why do you think the total of the four corner numbers is the same as four times the middle number?

Multiples of Tens and Tenths

Invite students to use a calculator to set up tables showing the number sentences and results of multiplying and dividing the same number by 10, 100, 1000 and so on, and by 0.1, 0.01, 0.001 and so on. Ask: Can you use the information to derive rules for multiplying and dividing by very large and very small numbers?

Relationships

Have students arrange materials to match the relationship between places in our numeration system. For example: Put out a small Base Ten Block, then the large 1000-cube. Ask: How many times bigger is the second cube than the first? What do you think the next-sized cube would look like? Do we have enough large cubes in the school to build the next-sized cube? How do you know? Do you have enough to build just the frame of the cube? Ask students to write the numbers for each of the cubes, and imagine what the fourth cube would look like. Ask: How do you say that number?

Calculator Experiment

Ask students to experiment with a calculator to work out the rule used on the calculator to produce patterns such as the following:
60, 600, 6000, 60 000, ...
60, 70, 80, 90, ...

Sort the patterns according to those in which you are constantly adding or subtracting and those in which you are constantly multiplying or dividing. Ask: How are they related to the way our number system is organized?

Grid Paper

Have students use 1-mm grid paper (See Appendix: Line Master 24) to show the relationship between places in our number system. In the top left-hand corner, students draw around a 1-mm square. Next, include the drawn square but draw around a square that is 100 times larger. Draw the third square 100 times larger than the second square. Record the numbers in each square. Estimate how much space you will need to make the fourth square. What number would that be? How do you know?

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Key Understanding 6

Some numbers have interesting or useful properties. Investigating the patterns in these special numbers can help us to understand them better.

Students should investigate properties of numbers and patterns associated with those properties. This should at least include odd and even, prime and composite, and square and cubic numbers.

When we partition collections and represent the result as a multiplication, we think of both the number of groups and the number in each group as factors of the original number. Thus, partitioning underpins the concept of "factor" and the classification of numbers into odd and even and composite and prime.

Students should develop the following understandings:

- Some collections can be shared into two equal groups (those with **even** numbers of items) and some cannot (those with **odd** numbers of items).
- If a collection can be shared into equal collections, then we say that the number of equal collections and the number in the equal collections are each **factors** of the number in the original collection. For example: 14 items can be shared into 14 collections of 1 item and 1 collection of 14 items, and into 7 collections of 2 items and 2 collections of 7 items. So we say that 14, 7, 2 and 1 are all factors of 14. 14 cannot be shared into collections of 3, so we say 3 is not a factor of 14. (Another name for factor is **divisor**.)
- Some collections (those with a **prime number** of items) can only be shared into smaller equal collections, if the smaller collections hold 1. So there are only two factors: one and the number.
- Some collections (those with a **composite number** of items) can be shared into smaller equal collections, with the smaller collections holding more than 1. So there are more than two factors.
- The number 1 has only one factor, which is 1. So the number one is neither prime nor composite.

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A closely related concept is that of **multiples**. A collection of 12 items can be thought of as a unit that is replicated to produce 12 items, 24 items, 36 items, 48 items, and so on. The numbers 12, 24, 36, 48, ... are then called the multiples of 12. Because of the inverse relationship between multiplication and division, we can, in turn, say that 12 is a factor of each of the numbers 12, 24, 36, 48 and so on.

Students should represent numbers in a wide variety of ways that bring out their properties. Thus, in Kindergarten to grade 3, students might investigate the occurrence of odd and even numbers and note that every second one is even and every second one is odd. In grades 3 to 5, they might note the pattern of digits in the units place for odd and even numbers, and use it to decide what side of the street houses will be on, or whether a particular number could be the solution to a problem. For example: Can 127 be one of the numbers in the sequence 4, 8, 12, 16, ...? In grades 5 to 8, they might use patterns to make a generalization about the effect of multiplying any number by an even number.

As an interesting variation on the idea that patterns are useful because they enable us to predict, students should also learn the reason that prime numbers are useful for cryptography—making secret codes—is because there is *no* pattern in the sequence of the prime numbers. Therefore, we cannot predict what will come next or later.

Around grade 7, students should begin to learn about the more common irrational numbers, in particular π (pi), and the idea of non-repeating decimals. Thus, they should be introduced to the idea that when you express π (pi) as a decimal, the sequence of digits after the decimal point goes on forever with no repetition or pattern (that is, it does not recur). This means we *cannot* write π (pi) exactly as a common fraction. They can also use their calculator as a tool for investigating terminating and recurring decimals, to find that both *can* always be written exactly as a common fraction. For example: $3.125 = 3\frac{1}{8}$ and $3.125125125125 \ldots = 3\frac{125}{999}$.

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Sample Learning Activities

K-Grade 3: ★ Introduction, Consolidation or Extension

Game Plans

Have students set out materials to represent players or use pairs of small name cards to plan rosters for using equipment or games that involve partners. Focus on the use of odd and even numbers of students who want to play. Ask students what to do when there is an odd number of players.

Handfuls

Ask students to identify odd and even numbers by playing *Handfuls*. Working with a partner, one student grabs a handful of straws or bottletops, then passes it to their partner. If the partner can share it out between the two people, they get a point. If not, the point goes to the other person. Ask students to record the numbers as they go. What do you notice about the numbers you can share?

Function Box

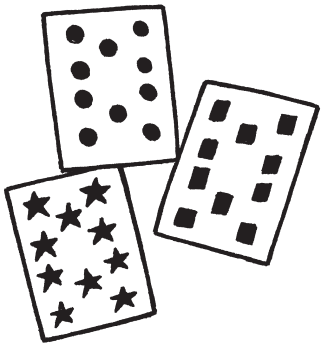
Extend the Function Box activities on page 247 by having a student write a number on a piece of paper and enter it into the box. Tell the students that the box is programmed to multiply by two. Ask: What numbers could come out? Will the number coming out be odd or even? How do you know?

Finding Groups

Have students use materials to work out which numbers from one to 12 can be made into equal groups. Ask them to make a chart showing the different groups for each number and refer to this to decide on a number when equal groups are needed.

Doubling

Invite students to investigate the built-in pattern when odd numbers are doubled. Ask them to create a card with ten spots, squares or stars on it. They cover some and say how many are left. They predict how many would be there if the spots were doubled and use a mirror to check. Ask: What has happened to the six dots? What will happen if you double five dots? When an odd number is doubled, do you get another odd number? Could you ever get an odd number when an even number is doubled?



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Odds and Evens

Have students describe numbers of things that occur naturally in odd and even groupings. For example: When making collage pictures or constructions of animals, insects or vehicles, each student selects their materials before beginning the picture or structure, and uses "odd" and "even" to describe the size of their collections. Ask: Have you got an even number of ears? Is that number of legs an odd or even number? Is one an odd or even number? How do you know which numbers are odd (or even)?

Diagrams

Following *Odds and Evens*, above, have students recall which numbers were odd and even and use diagrams to represent them.

Hands

Have two students face each other, then clap their hands three times before holding up between five and ten fingers. Have them show all the fingers on one hand and some extra fingers on the second hand. Together, students say how many fingers are held up altogether. Extend the activity by having students record whether the results are odd or even each time.



Halving

Ask students to look for patterns when halving. Start with a collection of, say, 12 jelly beans and halve then halve again. Record this as a sequence of numbers: 12, 6, 3. Start again with 24 jelly beans. Ask: Why do the same numbers appear?

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Sample Learning Activities

Grades 3-5: ★ ★ Important Focus

What's My Number?

Have each student make a set of 20 small blocks that are numbered from one to 20. Choose a number and give the students clues. For example: It is an even number. The students then remove the numbered blocks that it could not be. Continue with clues, such as: You say it when you count by 5s; When you write it, it has two places; It won't make three equal groups. Talk about the clues with the students after the game and ask: Which were the most helpful clues? Why? What did you need to know to answer the ... clue? Eventually, have a student help give clues, then have a student lead the game.

Constant Function

Ask students to use the constant function on a calculator to generate multiples of numbers up to ten. Using a 100-chart (See Appendix: Line Master 20), cover the numbers with different coloured counters and discuss the pattern each number makes. Have students look for overlap in patterns. For example: The fours pattern only uses numbers from the twos pattern. Ask: Why is this so? Look at the chart to see if any numbers are still exposed. Discuss why these numbers have not been covered (are they prime?).

Prime Numbers

Extend the *Handfuls* game on page 272 to get two points if the objects can be made into a 2 x ☐ rectangle, three points for a 3 x ☐ rectangle, and so on. Have students say which collections will not give them any points and why.

Twenty-Seven

Have students investigate whether the number 27 can be found by: adding two odd numbers, two even numbers, or an even and an odd number. Ask them to investigate other numbers and say if there is a pattern in the results. Extend to other operations.

Odd and Even Patterns

Ask students to investigate patterns associated with odd and even numbers. For example: Start with the number 19 and add six each time—what number sequence is created? Start with the number 89 and subtract 11 each time—what sequence is created? Have students predict whether the next number will be odd or even and test their predictions with the calculator.

Pyramids

Have students use blocks to build a pyramid. A pyramid one layer high uses 1 block, two layers high uses 4 blocks, three layers high uses 9 blocks. Students predict how many blocks are needed for the fifth, sixth and tenth layers.

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Investigating Primes

Invite students to investigate prime numbers by using blocks to design crates that will hold different numbers of bottles. The crates must be rectangular (including square), at least two bottles wide and hold up to 24 bottles. Ask students to record the crates made and the number of bottles they will hold. Ask: Can you make a crate to hold only seven bottles? Why not? Have the class identify the numbers that cannot be made into crates. These are prime numbers.

Times Tables Patterns

Have students investigate odd and even number patterns in the times tables. Ask: Are the answers to the two times table all even, all odd, or a combination of even and odd? Is there a pattern of odd and even numbers in the answers? Is there a pattern of even and odd numbers being multiplied? Look at all the other tables and work out a rule to say what happens when two even numbers are multiplied, two odd numbers are multiplied and an even and odd number are multiplied. Ask students to test out their rule: If you add an odd number to a multiple of nine, the result will be—sometimes odd, always odd, always even, never odd?

Shapes

Ask students to use shapes to investigate factors. For example: They take a handful of toothpicks, and decide how many triangles they can make. How many rectangles? Pentagons? Ask them to record the results using pictures and number sentences. Draw out the idea of factors, asking: Why is it that you get toothpicks left over for some shapes and not for others?

Multiples of Three

Have students build up multiples, using the inverse of the previous activity. For example: Make one triangle using three toothpicks, two using six toothpicks, and so on. Ask students to write the list of multiples of three next to their diagrams of the triangles to show how many toothpicks were used altogether.

Factors

Encourage students to use factors to solve problems. For example: We have 24 tomato plants to plant in a field. How could we arrange them into rectangular plots? How do you know you have found all possible shapes? How can you work out factors of numbers without using materials?

Is It a Multiple?

Have students use the constant function on the calculator to find multiples of a number and predict whether a given number will be a multiple. For example: Will 51 be a multiple of four? This can become a game by taking turns to predict the next number before pressing the = key.

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Sample Learning Activities

Grades 5-8: ★ ★ Important Focus

Equal Groups

Have students make different equal groups from the same quantity and record with a multiplication number sentence. For example: Make arrays with 36 squares. Ask: Can you make an array with seven in a row? Can you make an array with five rows? Why? How do you know if you have made all the arrays?

Venn Diagram

Ask students to construct a Venn diagram using three circles. Label these "divisible by 2", "divisible by 3" and "divisible by 5". Have students choose a two-digit number and decide in which section of the Venn diagram it belongs. They examine and describe the patterns formed in each section. Ask them to use the pattern to predict where a larger number belongs.

Factor Trees

Have students investigate factors and products by making different factor trees for the same number, such as 36:



Ask students to compare the factor trees. Ask: How do the numbers in the last row show that there are no more factors?

Investigate Factors

Invite different groups of students to carry out the following factor investigations:

- Classify the counting numbers according to the number of factors.
- On a 100-chart (See Appendix: Line Master 20), put counters of one colour on all the multiples of two, a different colour on multiples of three, four, five and so on up to ten. What do you notice about the height of the stacks of the counters?
- Select a number of square tiles. Record how many different rectangles you can make using all the tiles. Repeat for other numbers of tiles. How many different rectangles can you make with the squares? Do it for some more numbers. Put your findings together with others in your group and then organize the information.

At the completion of the three tasks, have students look for overall patterns in their classifications. Ask: What is the same about them? What is different?

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Square Numbers

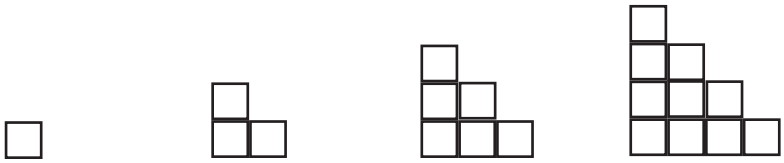
Have students use an activity like *Pyramids*, on page 274, to generate consecutive square numbers. Ask: What pattern can you see in the differences? How can you use this pattern to work out the square numbers that come after 121 and 144?

Areas of Squares

Extend *Square Numbers*, above, by presenting this scenario: When finding the areas of different sized squares, Miles recorded the areas along with the square numbers 1^2 ,2^2 , 3^2 and so on. He then noticed that the difference between the square numbers 2^2 and 3^2 was the same as 2 + 3, and the difference between 3^2 and 4^2 was the same as 3 + 4. Ask: Does this pattern work for larger consecutive square numbers? Is there a similar pattern in the differences between alternate square numbers? Why?

Square and Triangular Numbers

Investigate how square numbers are related to triangular numbers. Have students generate triangular numbers. (See *Triangular Numbers*, page 237.) For example: Build a set of steps with one block for the first step, three blocks for the second step, six blocks for the third step and so on.



List the first six triangular numbers, then add consecutive numbers. Ask: How can you prove and show that each square number is the sum of two triangular numbers? Is there a way of predicting which square number will result from the eighth and ninth triangles?

Fraction Patterns

Ask students to use a calculator to investigate patterns that occur when changing common fractions to decimal fractions. For example, enter 1 ÷ 7 = to change $\frac{1}{7}$ to a decimal fraction. Then change $\frac{2}{7}$ and $\frac{3}{7}$. Ask: What do you notice about the numbers? Can you predict the decimal fraction for $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$? Explore the patterns in decimal fractions for other common fractions.

Terminating Decimals

Have students predict which unit fractions will have a decimal that terminates and which ones will have a decimal that does not terminate. Ask them to use a calculator to classify the denominator from 2 to 20 according to whether or not they are factors of either 10, 100, 1000 and so on. Ask: If the denominator is a factor of a tens number, what kind of decimal do you get? If the denominator is not a factor of a tens number, what kind of decimal do you get?

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Appendix

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Planning Master	304
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Diagnostic Map Masters	312

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Name: _____ Date: _____

Line Master 1 **The Think Board**

Story	**Materials**
Picture/Diagram	**Number Sentence**

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Line Master 2 **The Think Board**

The think board					
Story	**Dramatization (Story picture)**	**Materials**	**Picture/ Diagram**	**Number sentence**	**Answer**

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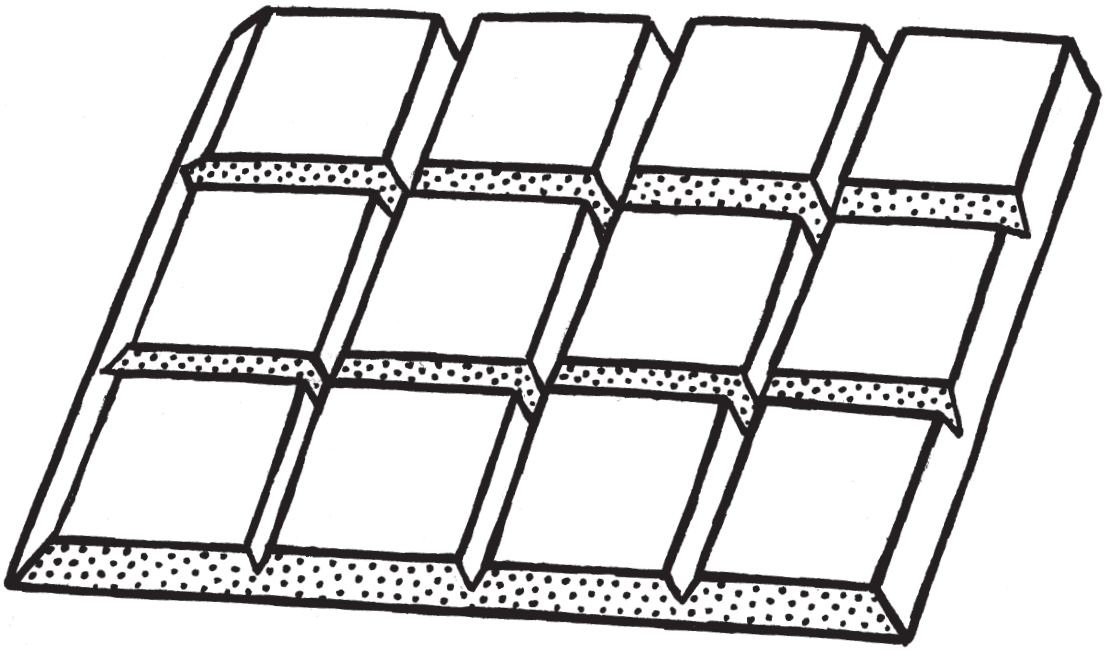
Line Master 3 **Division Think Board**

The think board				
Example	**Words**	**÷**	**)‾‾‾**	**Answer**

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Name: _____ Date: _____

Line Master 4 **Chocolate Bar to Share**



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Name: _____ Date: _____

Line Master 5 **Did You Know? 379 – 280**

Put a ✓ beside the problems for which 379 – 280 would give the answer. Give reasons for your choices.

- ❑ 379 students had to have their school photo taken. 280 students had been photographed by lunchtime. 80 of the students were in grade 6 and 200 were in kindergarten. How many students still have to be photographed?

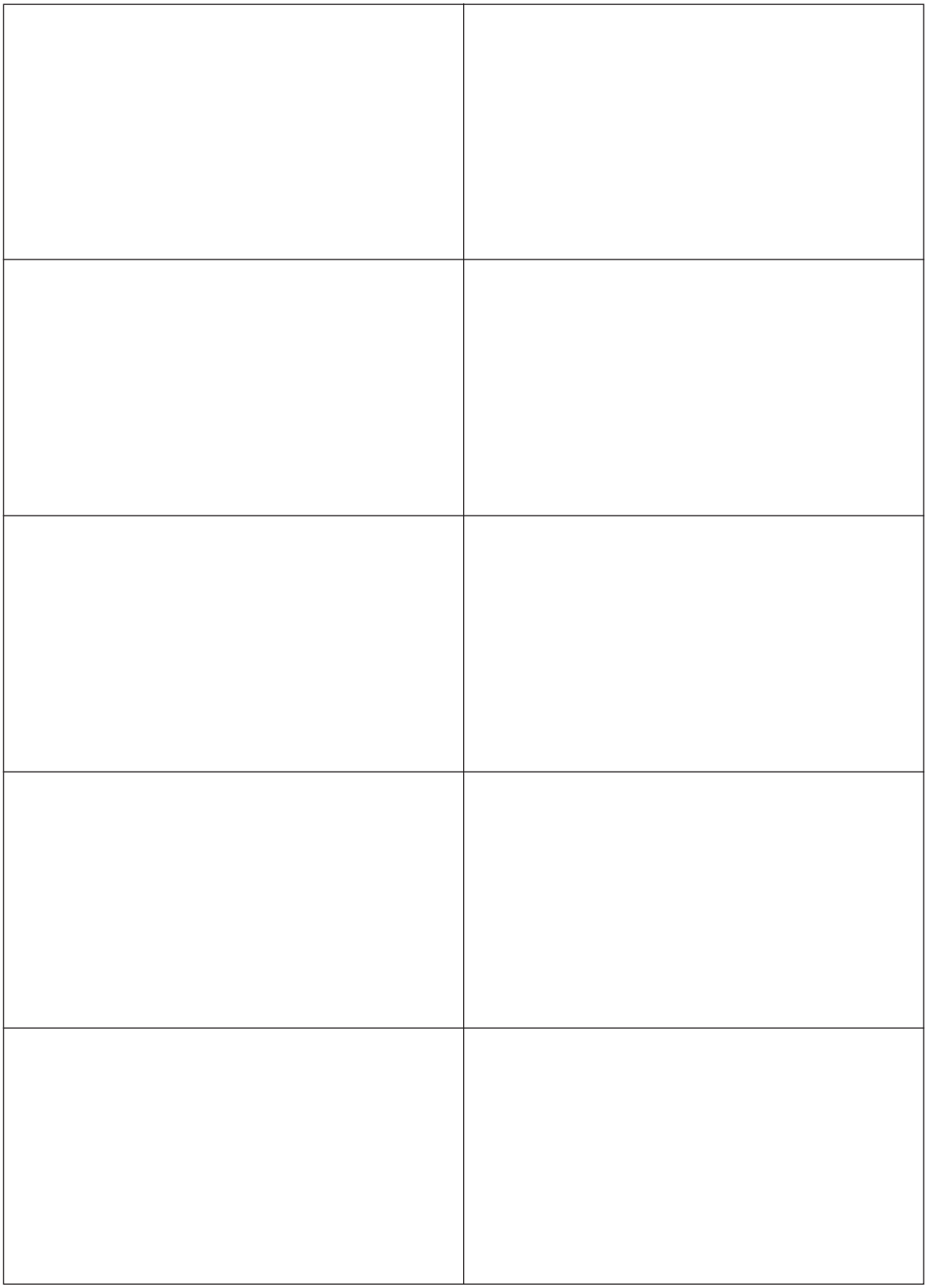
- ❑ There was 379 kg of dog food in the factory's freezer. 280 kg of it was sold to local shops. How much dog food was left over?

- ❑ 379 posts are needed for one fence and 280 posts are needed for another. How many posts are needed altogether?

- ❑ Maxine finished the car rally in 379 minutes. Her older sister, Jane, finished the rally in 280 minutes. How much longer did Maxine take?

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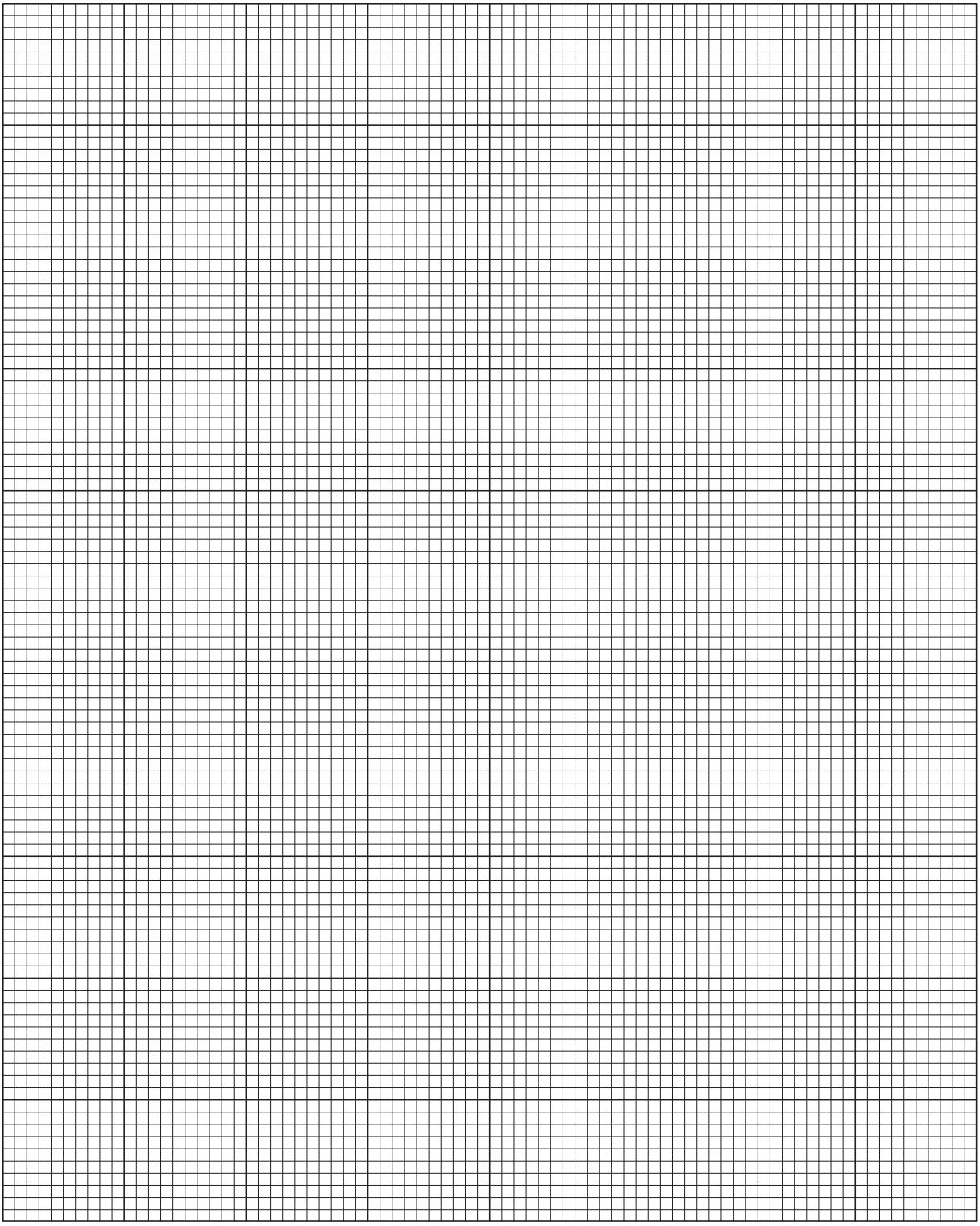
Line Master 6 **Ten-Frame**



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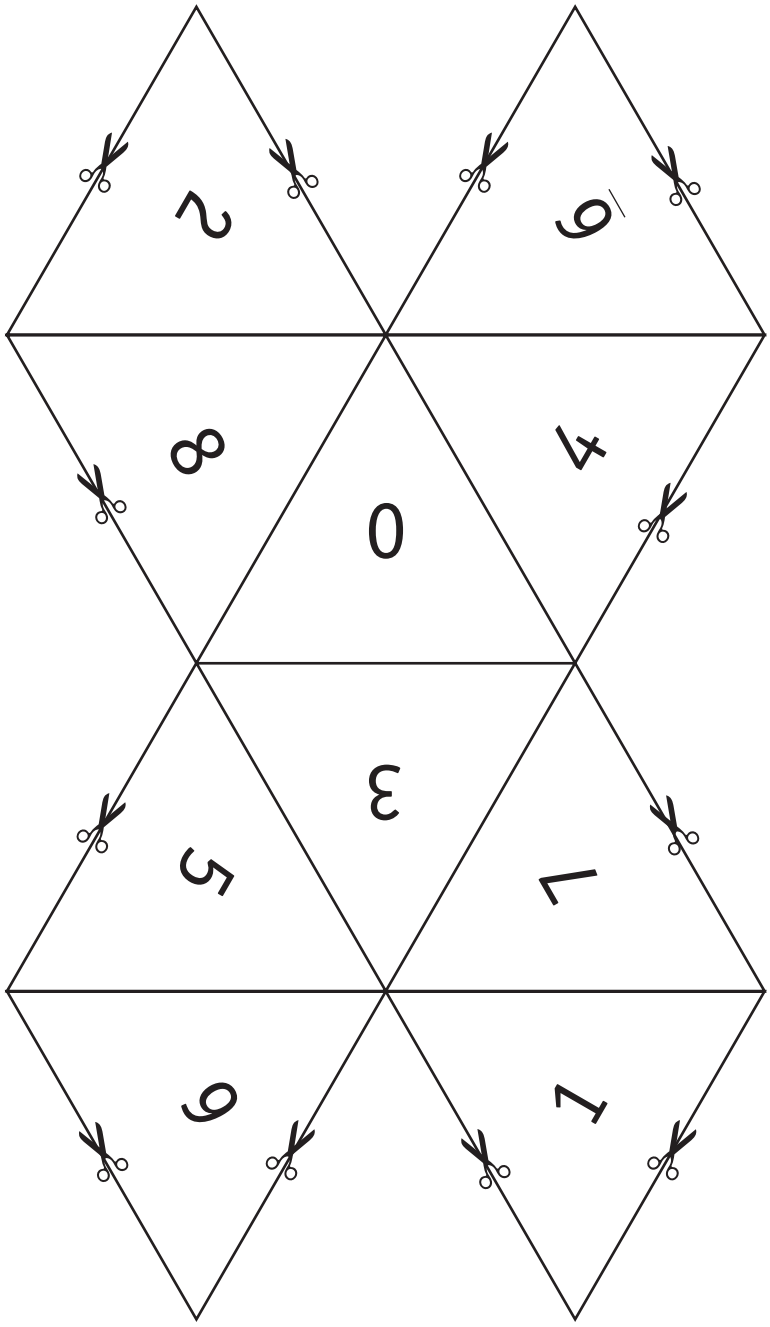
Line Master 7 **2-mm Grid Paper**



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Line Master 8 **Ten-Sided Number Cube**

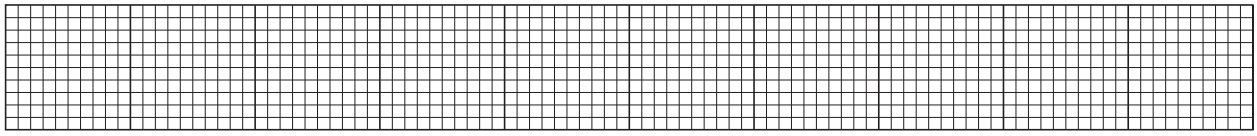
Cut out this shape, fold along the lines, and tape the edges together.

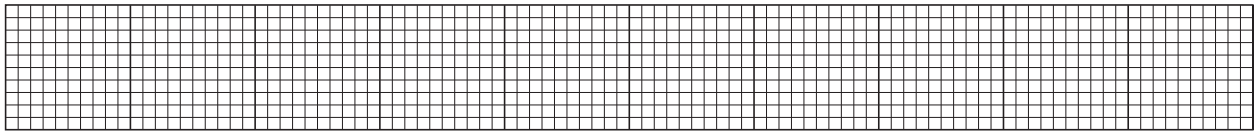


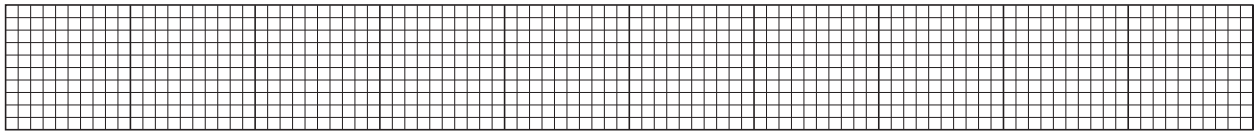
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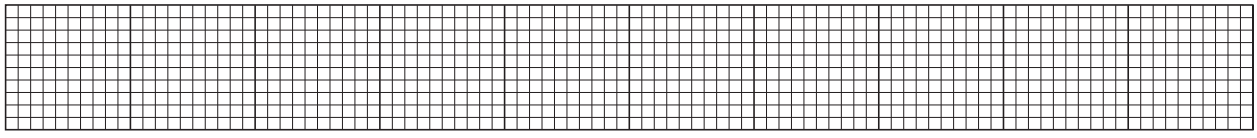
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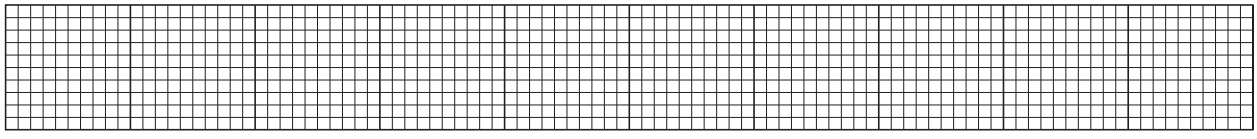
Line Master 9 **1000 Grids**

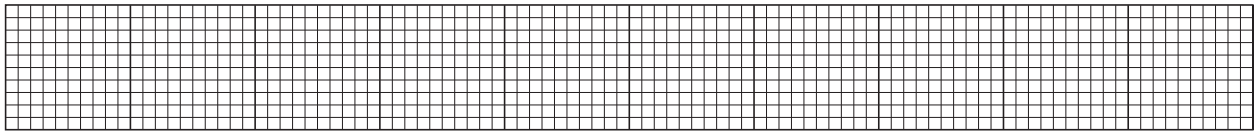








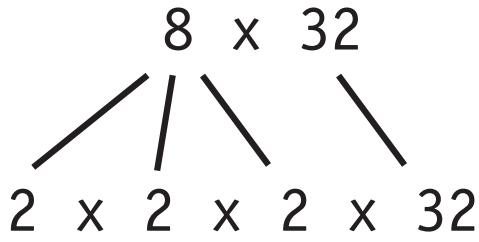




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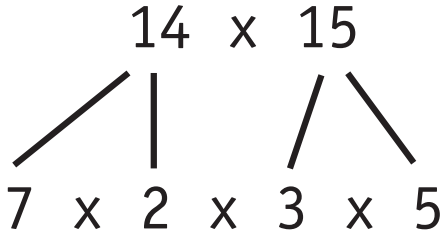
Line Master 10 **Easier Multiplication**

8 x 32 can be rewritten as 2 x 2 x 2 x 32.



That's double 32, three times.

14 x 15 can be rewritten as 7 x 2 x 3 x 5.

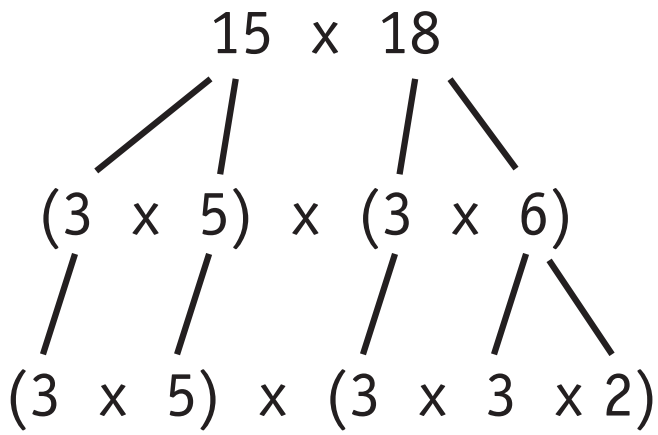


That's the same as 5 x 2 x 7 x 3, which is 10 x 21.

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Line Master 11 **Multiplying by Ten**

15 x 18 is the same as (3 x 5) x (3 x 6), which is the same as (3 x 5) x (3 x 3 x 2), which is the same as 3 x 3 x 3 x 10.

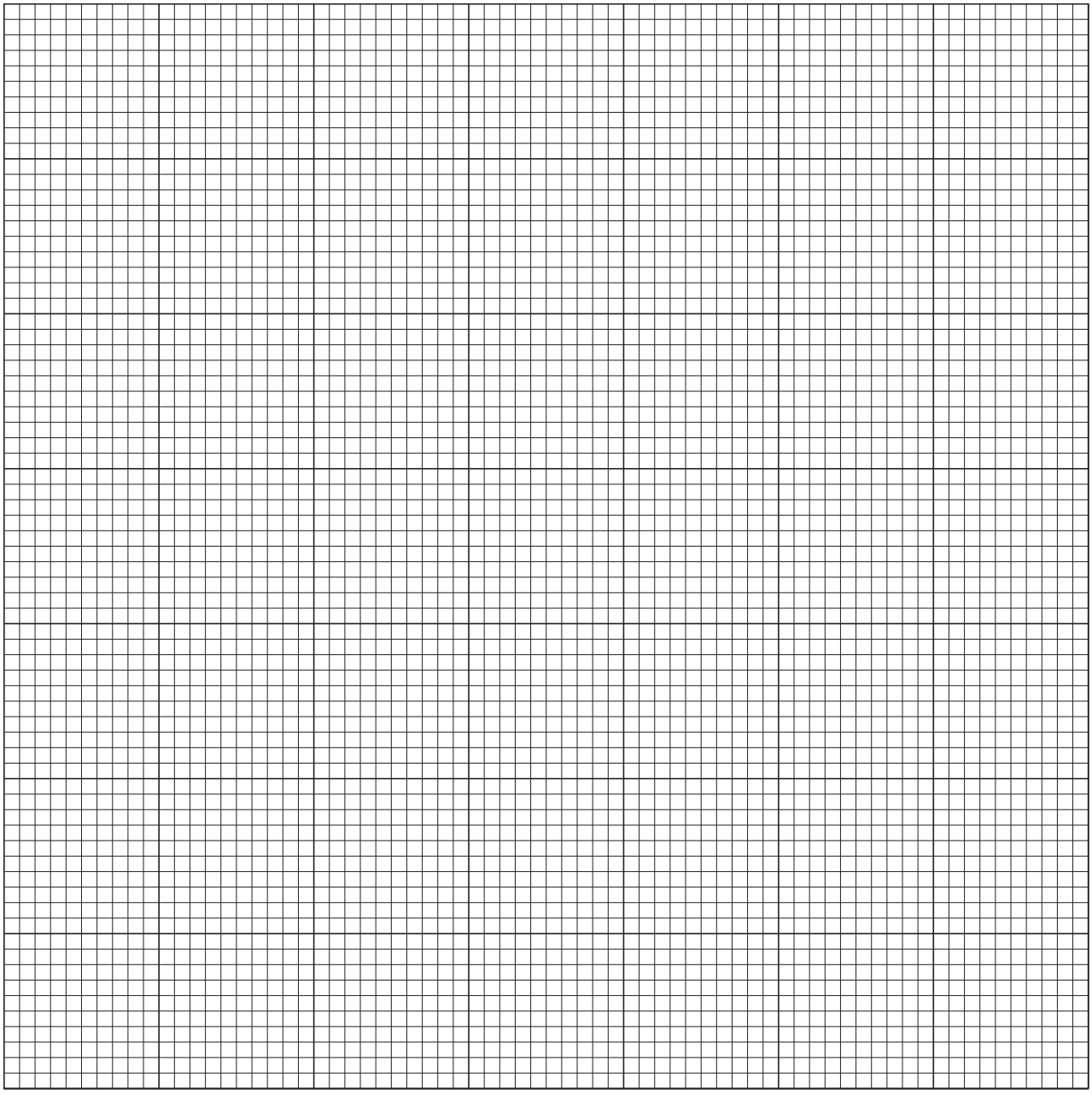


3 x 3 x 3 x 10

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Line Master 12 **Ten-Squared Grid Paper**



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Line Master 13 **Number Tiles**

1	2	3	4
5	6	7	8
9	10	11	12

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Line Master 14 **Number Cube Labels 10 to 60**

10	20
30	40
50	60

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Line Master 15 **Target Practice Number Cube Labels**

Labels for Number Cube 1:

0	1
2	3
4	5

Labels for Number Cube 2:

6	7
8	9
1	0

Labels for Number Cube 3:

10	10
10	1
1	1

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Line Master 16 **Multiply the Parts**

To multiply 99 x 6, think:

100 x 6 = 600

take away the extra 6:

600 – 6 = 594

How could this strategy help you to multiply 99 x 4?

99 x 8?

7 x 99?

Would the same strategy work if you multiply 19 x 3?

How?

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Line Master 17 **10 x 10 Array**

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Line Master 18 **Addition Facts**

Addition Facts

+	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

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Line Master 19 **Multiplication Facts**

Multiplication Facts

x	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

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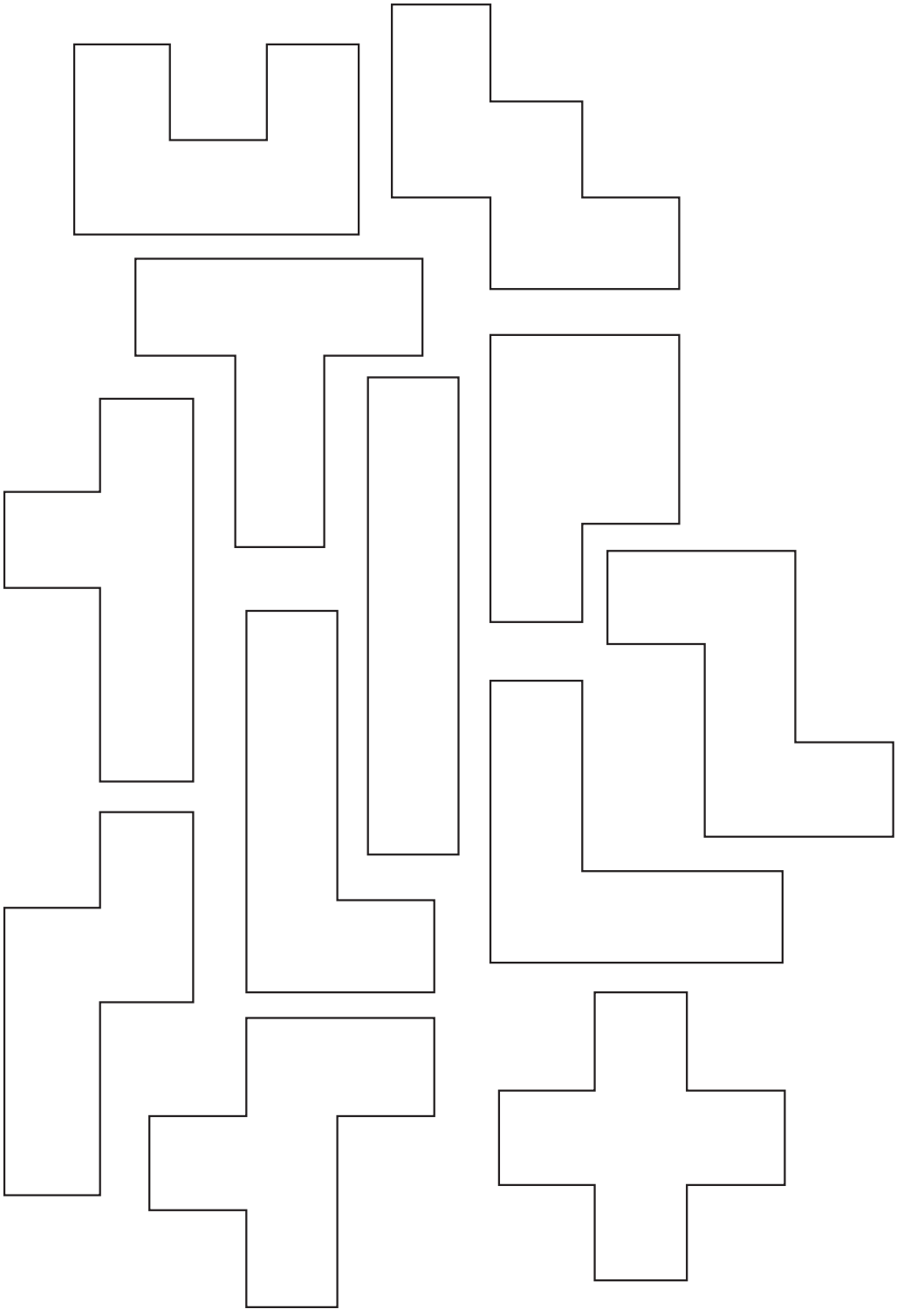
Line Master 20 **100-Chart**

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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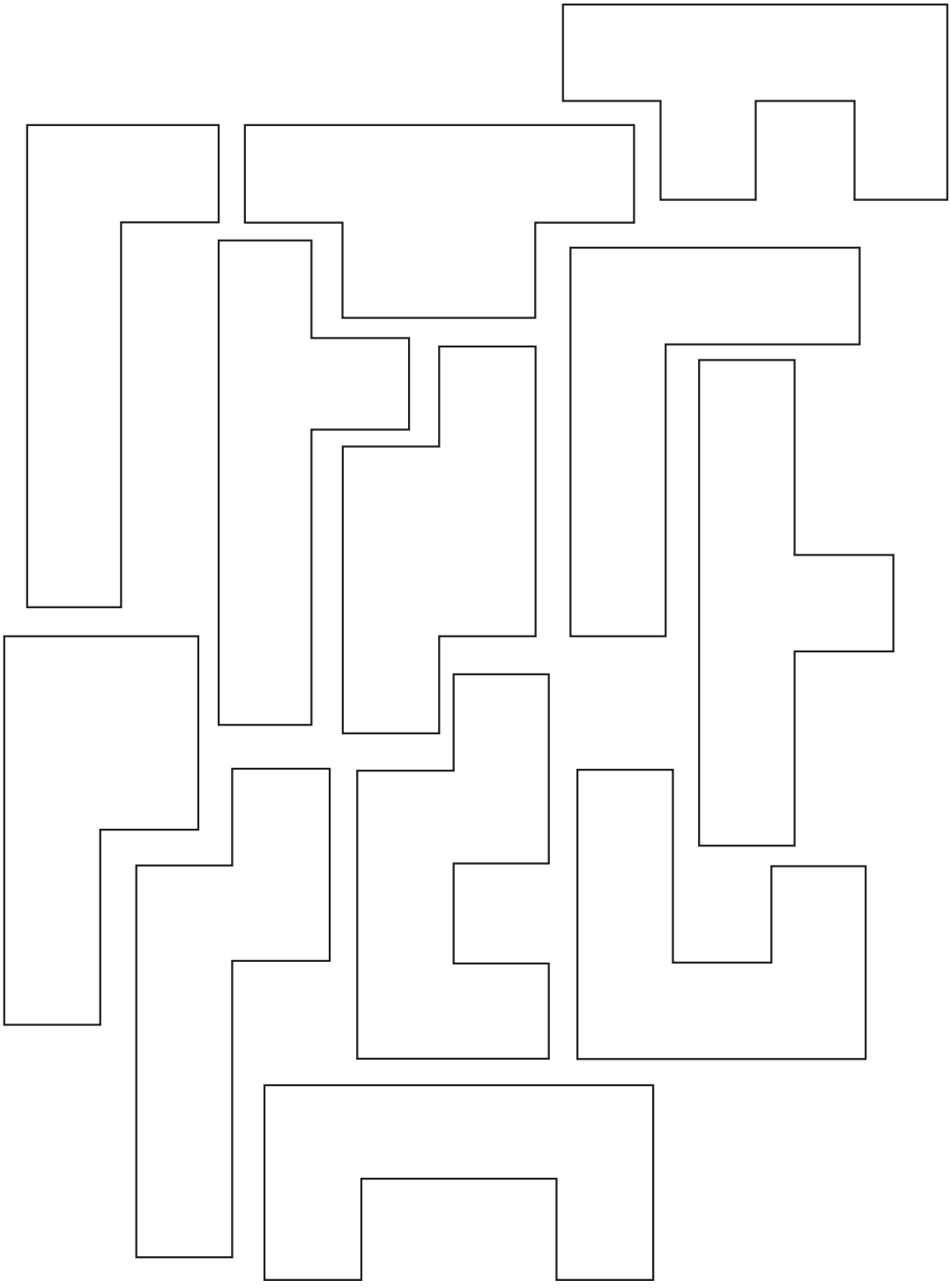
Line Master 21 **Pentonimoes**



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Line Master 22 **Hexominoes**



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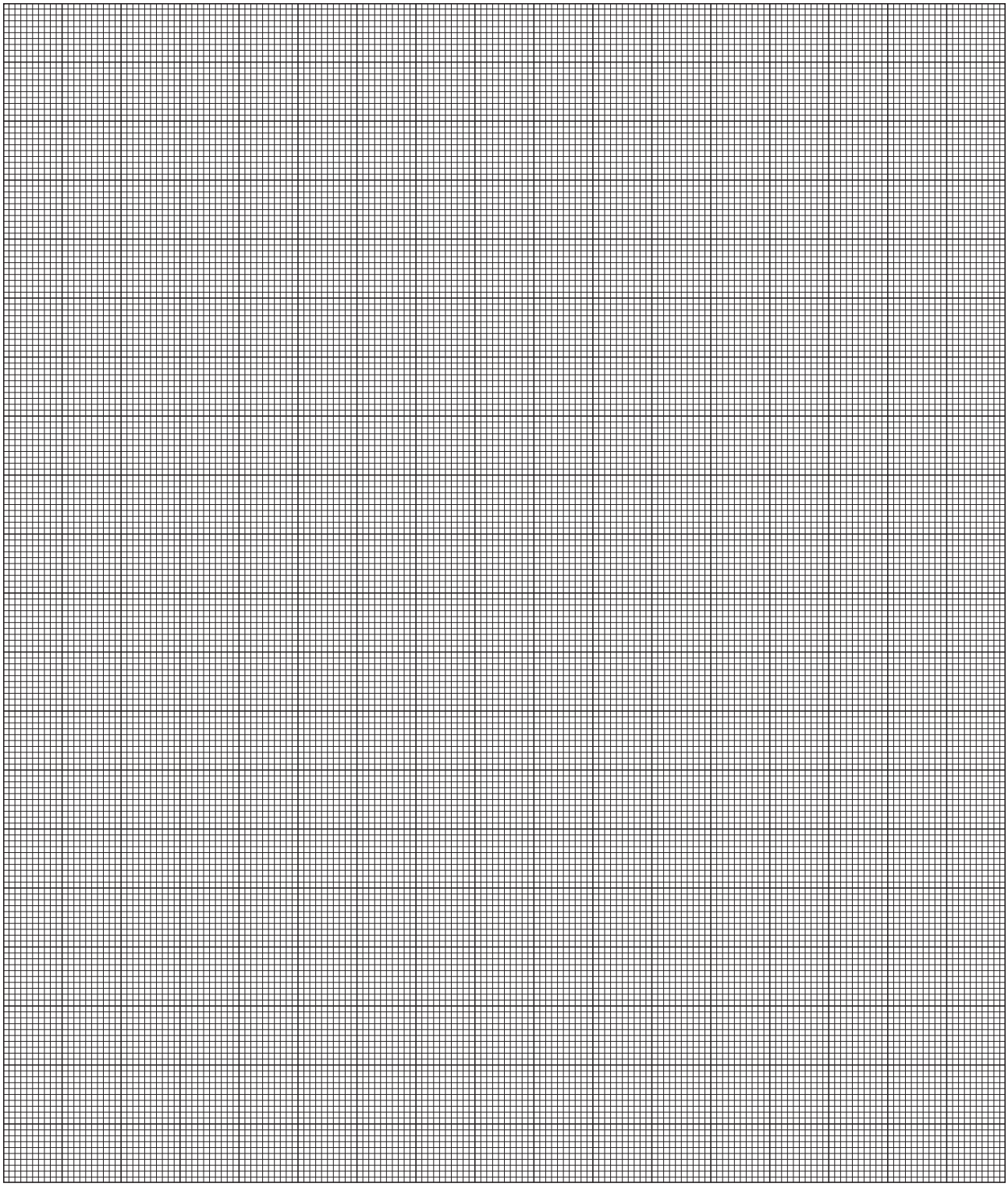
Line Master 23 **1000-Chart**

10	20	30	40	50	60	70	80	90	100
110	120	130	140	150	160	170	180	190	200
210	220	230	240	250	260	270	280	290	300
310	320	330	340	350	360	370	380	390	400
410	420	430	440	450	460	470	480	490	500
510	520	530	540	550	560	570	580	590	600
610	620	630	640	650	660	670	680	690	700
710	720	730	740	750	760	770	780	790	800
810	820	830	840	850	860	870	880	890	900
910	920	930	940	950	960	970	980	990	1000

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Line Master 24 **1-mm Grid Paper**



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Planning Master

Classroom Plan for Week _____, Term _____ **Grade Level: _____**

Curricular Goal/Key Understanding	Mathematical Focus	Activities	Focus Questions	Observations/ Anecdotes

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Tracking Master 1 **Ongoing Progress through the Number Diagnostic Map**

Record the date that students move into each developmental phase of the Number Diagnostic Map. A copy of this sheet can be placed in each student's math portfolio to chart individual growth over time.

Student Name	Emergent Phase	Matching Phase	Quantifying Phase	Partitioning Phase	Factoring Phase	Operating Phase

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Tracking Master 2 **Ongoing Progress through the Emergent Phase**

Record students' progress through the key indicators of the Emergent phase of Number and note the date students move to the Matching phase.

Student Name															
Emergent Phase															
▪ use "bigger", "smaller" and "the same" to describe differences...															
▪ anticipate whether an indicated change to a collection or quantity will make it bigger, smaller or leave it the same															
▪ distinguish spoken numbers from other spoken words															
▪ distinguish numerals from other written symbols															
▪ see at a glance how many are in small collections and attach correct number names to such collections															
▪ connect the differences they see between collections...															
▪ understand a request to share in a social sense and distribute items															
Moving to the Matching Phase															

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Tracking Master 3 **Ongoing Progress through the Matching Phase**

Record students' progress through the key indicators of the Matching phase of Number and note the dates students move from phase to phase.

Student Name															
Matching Phase															
Moved from the Emergent Phase															
▪ recall the sequence of number names at least into double digits															
▪ know how to count a collection…															
▪ understand that it is the last number said which gives the count															
▪ understand that building two collections by matching one to one leads to collections of equal size…															
▪ compare two collections one to one and use this to decide which is bigger and how much bigger															
▪ solve small number story problems…															
▪ share by dealing out an equal number of items or portions to each recipient…															
Moving to the Quantifying Phase															

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Tracking Master 4 **Ongoing Progress through the Quantifying Phase**

Record students' progress through the key indicators of the Quantifying phase of Number and note the dates students move from phase to phase.

Student Name													
Quantifying Phase													
Moved from the Matching Phase													
▪ without prompting, select counting as a strategy to solve problems…													
▪ use materials or visualize to decompose small numbers into parts…													
▪ find it obvious that when combining or joining collections counting on will give the same answer as starting at the beginning and counting the group													
▪ make sense of the notion that there are basic facts…													
▪ select either counting on or counting back for subtraction problems…													
▪ can think of addition and subtraction situations in terms of the whole and the two parts and which is missing													
▪ write number sentences that match how they think about the story line for… addition and subtraction problems													
▪ realize that repeated addition or skip counting will give the same result as counting by ones													
▪ realize that if they share a collection into a number of portions by… the portions must be equal…													
▪ understand that the more portions to be made from a quantity, the smaller the size of each portion													
Moving to the Partitioning Phase													

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Tracking Master 5 **Ongoing Progress through the Partitioning Phase**

Record students' progress through the key indicators of the Partitioning phase of Number and note the dates students move from phase to phase.

Student Name													
Partitioning Phase													
Moved from the Quantifying Phase													
▪ can compare whole numbers using their knowledge of the patterns…													
▪ make sense of why any whole number can be rewritten as the addition of other numbers													
▪ partition at least two- and three-digit numbers into standard component parts…													
▪ count up and down in tens from starting numbers like 23 or 79													
▪ write suitable number sentences for the range of addition and subtraction situations													
▪ use the inverse relationship between addition and subtraction to make a direct calculation…													
▪ can double count in multiplicative situations by representing one group and counting repetitions of that same group…													
▪ find it obvious that two different-shaped halves from the same size whole must be the same size													
▪ use successive splits to show that one-half is equivalent to 2 parts in 4…													
▪ partition a quantity into a number of equal portions to show unit fractions…													
Moving to the Factoring Phase													

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Tracking Master 6 **Ongoing Progress through the Factoring Phase**

Record students' progress through the key indicators of the Factoring phase of Number and note the dates students move from phase to phase.

Student Name													
Factoring Phase													
Moved from the Partitioning Phase													
▪ use their knowledge to generate alternative partitions													
▪ sustain a correct whole number place-value interpretation…													
▪ are flexible in their mental partitioning of whole numbers…													
▪ understand that a number can be decomposed and re-composed into its factors…													
▪ find it obvious that if 3 rows of 5 is 15, then both 15 divided by 3 and one-third of 15 are 5													
▪ can visualize an array to see, for example, that five blue counters is one-third of a bag of 15 counters…													
▪ visualize or draw their own diagrams to compare fractions with the same denominator…													
▪ use the idea of splitting a whole into parts to understand, for example, that 2.4 is $2 + \frac{4}{10}$ …													
▪ relate fractions and division knowing, for example, that $\frac{3}{4}$ can be thought of as $3 \div 4$…													
▪ know that they can choose between multiplication or division to make calculating easier													
▪ understand why grouping and sharing problems can be solved by the same division process													
▪ interpret multiplication situations as "times as much" and so can see that 12 is 3 times as much as 4, and 8 is 10 times smaller than 80													
▪ select an appropriate multiplication or division operation on whole numbers…													
▪ can see why multiplication of whole numbers is commutative…													
Moving to the Operating Phase													

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Tracking Master 7 **Ongoing Progress through the Operating Phase**

Record students' progress through the key indicators of the Operating phase of Number and note the dates students move from phase to phase.

Student Name													
Operating Phase													
Moved from the Factoring Phase													
▪ represent common and decimal fractions both smaller and greater than 1 on a number line													
▪ generalize their understanding of whole number place value to include the cyclical pattern beyond the thousands...													
▪ use their understandings of the relationship between successive places to order decimal numbers...													
▪ use the cyclical pattern in the places to count forwards and backwards in tenths, hundredths, thousandths...													
▪ are flexible in partitioning decimal numbers													
▪ realize that for multipliers smaller than 1, multiplication makes smaller, and for divisors smaller than 1, division makes bigger													
▪ select an appropriate number of partitions to enable a quantity to be shared into two different numbers of portions													
▪ construct successive partitions to model multiplication situations													
▪ produce their own diagrams to compare or combine two fractions, ensuring that both fractions are represented on identical wholes													
▪ split and recombine fractions visually or mentally to add or subtract													
▪ recognize the need to multiply in situations where the multiplier is a fractional number													
▪ can write suitable number sentences for the full range of multiplication and division situations, involving whole numbers, decimals and fractions													

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Emergent Phase

During the Emergent Phase

Students reason about small amounts of physical materials, learning to distinguish small collections by size and recognizing increases and decreases in them. They also learn to recognize and repeat the number words used in their communities and to distinguish number symbols from other symbols. There is a growing recognition of what is the same about the way students' communities use numbers to describe collections and what is different between collections labelled with different numbers.

As a result, students come to understand that number words and symbols can be used to signify the "numerosity" of a collection.

By the end of the Emergent phase, students typically:

- use "bigger", "smaller" and "the same" to describe differences between small collections of like objects and between easily compared quantities
- anticipate whether an indicated change to a collection or quantity will make it bigger, smaller or leave it the same
- distinguish spoken numbers from other spoken words
- distinguish numerals from other written symbols
- see at a glance how many are in small collections and attach correct number names to such collections
- connect the differences they see between collections of one, two and three with the number string: "one, two, three, ..."
- understand a request to share in a social sense and distribute items or portions

These students recognize that numbers may be used to signify quantity.

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Matching Phase

Most students will enter the Matching phase between 3 and 5 years of age.

As students move from the Emergent phase to the Matching phase, they:

- may actually see at a glance how many there are in a small collection, such as six pebbles, yet may not be able to say the number names in order
- may say a string of the number names in order (*one, two, three, four, …*), but not connect them with how many are in collections
- may be beginning to see how to use the number names to count, but may get the order of the names wrong
- can tell by looking which of two small collections is bigger; however, they generally cannot say how much bigger
- may distribute items or portions in order to "share", but may not be concerned about whether everyone gets some, the portions are equal, or the whole amount is used up

During the Matching Phase

Students use numbers as adjectives that describe actual quantities of physical materials. Through stories, games and everyday tasks, students use one-to-one relations to solve problems where they can directly carry out or imagine the actions suggested in the situation. They learn to fix small collections to make them match, "deal out" collections or portions, and to respect the principles of counting.

As a result, students learn what people expect them to do in response to requests such as: How many are there? Can you give me six forks? How many are left? Give out one (two) each. Share them.

By the end of the Matching phase, students typically:

- recall the sequence of number names at least into double digits
- know how to count a collection, respecting most of the principles of counting
- understand that it is the last number said which gives the count
- understand that building two collections by matching one to one leads to collections of equal size, and can "fix" one collection to make it match another in size
- compare two collections one to one and use this to decide which is bigger and how much bigger
- solve small number story problems which require them to add some, take away some, or combine two amounts by imagining or role playing the situation and counting the resulting quantity
- share by dealing out an equal number of items or portions to each recipient, cycling around the group one at a time or handing out two or three at a time

These students use one-to-one relations to share and count out.

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Quantifying Phase

Most students will enter the Quantifying phase between 5 and 6 or more years of age.

As students move from the Matching phase to the Quantifying phase, they:

- often do not spontaneously use counting to compare two groups in response to questions, such as: Are there enough cups for all students?
- may "skip count" but do not realize it gives the same answer as counting by ones and, therefore, do not trust it as a strategy to find how many
- often still think they could get a different answer if they started at a different place, so do not trust counting on or counting back
- often can only solve addition and subtraction problems when there is a specific action or relationship suggested in the problem situation which they can directly represent or imagine
- have difficulty linking their ideas about addition and subtraction to situations involving the comparison of collections
- may lay out groups to represent multiplicative situations, but do not use the groups to find out how many altogether, counting ones instead
- may represent division-type situations by sharing out or forming equal groups, but become confused about what to count to solve the problem, often choosing to count all the items
- may deal out an equal number of items or portions in order to share, but do not use up the whole quantity or attend to equality of the size of portions
- often do not realize that if they have shared a quantity, then counting one share will also tell them how many are in the other shares
- may split things into two portions and call them halves but associate the work "half" with the process of cutting or splitting and do not attend to equality of parts

During the Quantifying Phase

Students reason about numerical quantities and come to believe that if nothing is added to, or removed from, a collection or quantity, then the total amount must remain the same even if its arrangement or appearance is altered.

As a result, students see that the significance of the number uttered at the end of the counting process does not change with rearrangement of the collection or the counting strategy. They interpret small numbers as compositions of other numbers.

Also as a result, they develop the idea that constructing fair shares requires splitting the whole into equal parts without changing the total quantity and so begin to see the part-whole relations that link sharing and fractions.

See over ⇨

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Quantifying Phase cont.

By the end of the Quantifying phase, students typically:

- without prompting, select counting as a strategy to solve problems, such as: Are there enough cups? Who has more? Will it fit?
- use materials or visualize to decompose small numbers into parts empirically; *8 is the same as 5 with 3*
- find it obvious that when combining or joining collections counting on will give the same answer as starting at the beginning and counting the group
- make sense of the notion that there are basic facts, such as 4 + 5 is always 9, no matter how they work it out or in what arrangement
- select either counting on or counting back for subtraction problems, depending on which strategy best matches the situation
- can think of addition and subtraction situations in terms of the whole and the two parts and which is missing
- write number sentences that match how they think about the story line (semantic structure) for small number addition and subtraction problems
- realize that repeated addition or skip counting will give the same result as counting by ones
- realize that if they share a collection into a number of portions by dealing out or continuous halving and use up the whole quantity, then the portions must be equal regardless of how they look
- understand that the more portions to be made from a quantity, the smaller the size of each portion

These students use part-part-whole relations for numerical quantities.

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Partitioning Phase

Most students will enter the Partitioning phase between 6 and 9 years of age.

As students move from the Quantifying phase to the Partitioning phase, they:

- often cannot decompose into parts numbers that they cannot visualize or represent as quantities, so have difficulty in partitioning larger numbers to make calculation easier; for example, students need to count forwards or backwards by ones to find the difference between 25 and 38
- often use strategies based on materials, counting on or counting back to solve addition and subtraction problems, but do not link these strategies or different problem types to a single operation (either + or -)
- may be unable to use the inverse relationship between addition and subtraction to choose the more efficient of counting on or counting back for solving particular problems
- often write their number sentences after they have solved the problem with materials, counting or basic facts, so they may be unable to write number sentences in advance when needed for problems involving larger numbers
- can count equal groups by physically or mentally laying out each group, but think of and treat each group as distinct from the others
- often believe that for two halves there must be exactly two pieces; for example, students may deny the equality of one-half and two-quarters unless the two-quarters are "stuck back together"
- although understanding that the two halves they have formed by dealing out or splitting must be equal, may think that a half formed one way could be bigger than a half formed another way
- may ignore the size of portions when choosing fraction names; for example, describing one part in seven as one-seventh regardless of whether the seven portions are equal
- often do not link sharing to unit fractions and may think that eighths are bigger than thirds because 8 is bigger than 3

During the Partitioning Phase

Students come to see the significance of whole numbers having their own meaning independent of particular countable objects. They learn to use part-whole reasoning without needing to see or visualize physical collections.

As a result, students see that numbers have magnitudes in relation to each other, can interpret any whole number as composed of two or more other numbers, and see the relationship between different types of addition and subtraction situations.

Also as a result, students see that numbers can be used to count groups and that they can use one group as a representative of other equal groups. They trust, too, that appropriate partitioning of quantities must produce equal portions.

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Partitioning Phase cont.

By the end of the Partitioning phase, students typically:

- can compare whole numbers using their knowledge of the patterns in the number sequence, and think of movements between numbers without actually or mentally representing the numbers as physical quantities
- make sense of why any whole number can be rewritten as the addition of other numbers
- partition at least two- and three-digit numbers into standard component parts (*326 = 300 + 20 + 6*) without reference to actual quantities
- count up and down in tens from starting numbers like 23 or 79
- write suitable number sentences for the range of addition and subtraction situations
- use the inverse relationship between addition and subtraction to make a direct calculation possible; for example, re-interpret 43 – 27 as "what do you have to add to 27 to get 43" and so count on by tens and ones
- can double count in multiplicative situations by representing one group (*by holding up four fingers*) and counting repetitions of that same group, simultaneously keeping track of the number of groups and the number in each group
- find it obvious that two different-shaped halves from the same size whole must be the same size and are not tricked by perceptual features
- use successive splits to show that one-half is equivalent to 2 parts in 4, 4 parts in 8, and so on and expect that if the number of portions is doubled, they halve the size of each portion
- partition a quantity into a number of equal portions to show unit fractions and, given a particular quantity, will say that one-third is more than one-quarter

These students use additive thinking to deal with many-to-one relations.

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Factoring Phase

Most students will enter the Factoring phase between 9 and 11 years of age.

As students move from the Partitioning phase to the Factoring phase, they:

- can "work out" a non-standard partition (*47 - 30 = 17*), but they may not see it as following automatically from the way numbers are written
- often do not realize that the digit in the tens (hundreds) place refers to groups of ten (hundred) even when they correctly use the labels "ones", "tens" and "hundreds"
- have developed ideas about decimals based on daily use for money and measures, so may think the decimal point separates two whole numbers, where the whole numbers refer to different-sized units; for example, when referring to money, they may read 6.125 as if the 6 is dollars and the 125 is cents and thus "round" it to $7.25 or say that 6.125 > 6.25
- may rightly think of decimals as another way to represent fractional numbers but, for example, think 0.6 is one-sixth
- often write related divisions and multiplications (*6 x 3 = 18, 18 ÷ 3 = 6, 18 ÷ 6 = 3*) by working each out, are unable to use the inverse relationship between division and multiplication to work out an unknown quantity
- may not understand why grouping can be used to solve a sharing problem
- can write multiplication number sentences for problems which they can think of as "groups of", but may solve other types of multiplicative problems only with materials or by counting
- do not understand why multiplication is commutative; for example, they often do not see that four piles of 13 must be the same amount as 13 groups of 4
- may believe that to show a fraction of a collection the denominator must match the total number of items and will be unable, for example, to recognize six parts in 18 as one-third
- may think of $\frac{1}{3}$ only as one part out of a collection or quantity which has been split into three equal parts, but do not also recognize it as one in each three
- may think of fractions as quantities rather than numbers and not see the significance of using the same unit as the basis for comparing fractions, so do not see why $\frac{1}{3}$ must be bigger than $\frac{1}{4}$
- may see fractions, such as three-quarters, literally as three pieces each of one-quarter and will not accept one piece which is three-quarters of the whole

During the Factoring Phase

Students extend their additive ideas about whole numbers to include the coordination of two factors needed for multiplicative thinking. They learn to construct and coordinate groups of equal size, numbers of groups and a total amount. Students also learn to visualize multiplicative situations in terms of a quantity arranged in rows and columns (an array).

See over

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Factoring Phase cont.

As a result, students see the significance of the connection between groups of ten or groups of one hundred and the way we write whole numbers. They are able to relate different types of multiplication and division situations involving whole numbers. They also link the ideas of repeating equal groups, splitting a quantity into equal parts and fractions.

By the end of the Factoring phase, students typically:

- use their knowledge to generate alternative partitions, for example, the 2 in the tens place in 426 refers to 2 groups of 10
- sustain a correct whole number place-value interpretation in the face of conflicting information
- are flexible in their mental partitioning of whole numbers, confident that the quantity has not changed
- understand that a number can be decomposed and re-composed into its factors in a number of ways without changing the total quantity
- find it obvious that if 3 rows of 5 is 15, then both 15 divided by 3 and one-third of 15 are 5
- can visualize an array to see, for example, that five blue counters is one-third of a bag of 15 counters, both because 15 can be split into three parts each of five and one in every three counters will be blue
- visualize or draw their own diagrams to compare fractions with the same denominator ($\frac{3}{7}$ and $\frac{5}{7}$) or simple equivalences ($\frac{1}{4}$ and $\frac{2}{8}$)
- use the idea of splitting a whole into parts to understand, for example, that 2.4 is $2 + \frac{4}{10}$ and 2.45 is $2 + \frac{45}{100}$
- relate fractions and division knowing, for example, that $\frac{3}{4}$ can be thought of as $3 \div 4$ and 3 things shared among 4 students has to be $\frac{3}{4}$
- know that they can choose between multiplication or division to make calculating easier
- understand why grouping and sharing problems can be solved by the same division process
- interpret multiplication situations as "times as much" and so can see that 12 is 3 times as much as 4, and 8 is 10 times smaller than 80
- select an appropriate multiplication or division operation on whole numbers including for problems that are not easily interpreted as "groups of"; for example, combination and comparison problems
- can see why multiplication of whole numbers is commutative; for example, knowing without calculating, that 4 piles of 9 objects must be the same amount as 9 piles of 4 objects

These students think both additively and multiplicatively about numerical quantities.

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Operating Phase

Most students will enter the Operating phase between 11 and 13 years of age.

As students move from the Factoring phase to the Operating phase, they:

- often continue to rely largely on their knowledge of the "named" places in reading and writing numbers, so have difficulty writing numbers with more than four digits
- may label the places to the right of the decimal point as tenths and hundredths and write 2.45 as $2 + \frac{4}{10} + \frac{5}{100}$, for example, but cannot link this with other ways of writing the decimal, such as: $2 + \frac{45}{100}$
- may think decimals with two places are always hundredths and write 2.45 as $2 + \frac{45}{100}$, but do not link this with the pattern in whole-number place value and so do not see 2.45 as $2 + \frac{4}{10} + \frac{5}{100}$
- often are unable to select a common partitioning (denominator) to enable two fractions to be compared or combined unless an equivalence they already know is involved
- often ignore the need to draw two fractions on identical wholes in order to compare or combine them
- may be unable to select an appropriate operation in situations where they cannot think of the multiplier or divisor as a whole number
- may resist selecting division where the required division involves dividing a number by a bigger number
- often believe that multiplication "makes bigger" and division "makes smaller"

During the Operating Phase

Students learn to interpret multipliers as "times as much as" or "of" rather than simply counters of groups, so can think of them as "operators" that need not be whole numbers. Students also come to see that any number can be thought of as a unit which can be repeated or split up any number of times.

As a result, students see how the intervals between whole numbers can be split and re-split into increasingly smaller intervals and realize the significance of the relationship between successive places. For example, the value of each place is ten times the value of the place to its right and one-tenth of the value of the place to its left.

Also as a result, students learn to make multiplicative comparisons between numbers, deal with proportional situations, and integrate their ideas about common and decimal fractions.

By the end of the Operating phase, students typically:

- represent common and decimal fractions both smaller and greater than 1 on a number line
- generalize their understanding of whole-number place value to include the cyclical pattern beyond the thousands, so can read, write and say any whole numbers

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Operating Phase cont.

- use their understanding of the relationship between successive places to order decimal numbers regardless of the number of places
- use the cyclical pattern in the places to count forwards and backwards in tenths, hundredths, thousandths, including up and over whole numbers
- are flexible in partitioning decimal numbers
- realize that for multipliers smaller than 1, multiplication makes smaller, and for divisors smaller than 1, division makes bigger
- select an appropriate number of partitions to enable a quantity to be shared into two different numbers of portions; such as 5 or 3
- construct successive partitions to model multiplication situations; *I took half the cake home and then ate one-third of it*
- produce their own diagrams to compare or combine two fractions, ensuring that both fractions ($\frac{2}{3}$ and $\frac{1}{4}$) are represented on identical wholes
- split and recombine fractions visually or mentally to add or subtract; $\frac{1}{2} + \frac{1}{4}$ is $(\frac{1}{4}, \frac{1}{4}) + \frac{1}{4} = \frac{3}{4}$
- recognize the need to multiply in situations where the multiplier is a fractional number
- can write suitable number sentences for the full range of multiplication and division situations involving whole numbers, decimals and fractions

These students can think of multiplications and divisions in terms of operators.

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