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Last One Standing:

Creative, Cooperative Problem Solving

David A. Green

Eyes darted around the circle as some students just followed along and others counted ahead, trying to figure out if they would survive or not. "In," said one student. "Out," said the next, and then sat down. So the game went, around and around the circle, with every other player sitting out until only one third grader remained standing—the winner.

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The Game

I had gathered my students to play a favorite counting game, in which the winner would receive an imaginary prize. After a spirited discussion, the class decided that the winner of the contest would be given his or her favorite food for lunch for the rest of the school year and would have double science time and free time, two classroom favorites. I asked the students to raise their hands if they were interested in entering the contest. Everybody raised her or his hand. "But there is a catch," I continued. "Anyone who enters the contest and loses—which is everybody except the one winner—will lose all of his or her science and free time. Who still wants to enter the contest?" Far fewer hands went up. "The goal," I concluded, "is to figure out how to win every time."

Since the beginning of the school year, I had emphasized problem solving and number sense in my mathematics curriculum, impressing on my students how vital these skills are to good mathematicians. Daily "math challenges" covering a range of content featured problems for which the method of solving was not defined. The students had encountered challenges related to our studies of place value, addition and subtraction, geometry, measurement, multiplication, and probability. Students had to figure out how to solve such challenges by drawing on a variety of problem-solving strategies and using their reasoning, logic, and mathematics skills. The activities gave us an opportunity to discuss, and reflect on, the process of doing mathematics. They also underscored the fact that there is not one "right way" to solve a problem and helped my students take risks as learners. Having spent much of the year developing their problem-solving and number sense skills, my students needed to tackle a larger challenge, one that would weave together some of the mathematics strands they had already studied and would require them to collect, record, organize, and analyze data. They also needed a challenge that would require the use of not a single problem-solving strategy but multiple strategies. Thus my third graders embarked on a week-long mathematics exploration that blended together problem solving and number sense, cooperative and independent learning, and a range of mathematics Content and Process Standards set forth in *Principles and Standards for School Mathematics* (NCTM 2000).

The game that I developed is played as follows: The contestants stand in a circle. The first person in the circle says "In." The next person says "Out," sits down and is out of the game. The next person says "In," and the next player sits down. The contest continues repeatedly around the circle, knocking every other person out until only one survivor remains. A similar activity, "King Arthur's Problem," is presented in Marilyn Burns' book *Math For Smarty Pants*, in which knights risk their lives vying for the princess's hand in marriage (Burns 1982).

Our game started with eleven students in the circle and the rest looking on. I identified the starting person as "Player 1" and asked students to predict who would win. Some students chose the winner randomly and others did a quick count around the circle. After everyone made a prediction, we started the elimination process until we had our winner: Player 7.

Before starting another round, I added an extra student to the circle and asked the class to make new predictions. Based on the results of the first trial, some students stuck with Player 7, while others chose a new winner. Many chose Player 8, the person following the first winner. The students reasonably, but wrongly, assumed that adding one more person to the circle would increase the winning spot by one, from Player 7 to Player 8, and many students were surprised when Player 9 won the round. Nevertheless, many in the class noted that they now knew where to stand if eleven or twelve people played the game.

"There is an even bigger catch," I next informed the class. "When you commit to being in the contest, you don't know how many other people will be in the circle. Therefore, you can't count on knowing the one position in which to stand. For our third-grade contest, there can be 1 through 25 people in the circle, and you won't know how many people are playing until just before the contest begins. You need to be prepared for a circle of any size. Although the contest is currently limited to third graders, it is possible that others might be allowed to enter, bringing the number of players into the hundreds or thousands."

Making a Chart

Before they could consider that scenario, however, the students had to determine the winning positions for games with 1 to 25 players. Back at our seats, we discussed how we might solve these problems without actually playing the games. I asked the class to think about the different problem-solving methods we previously had used in class. Students quickly mentioned the possibility of making a diagram, in which they would use pencil and paper to draw a circle, write numbers around it, and then cross out every other number until one was left. The students began to work on the problems with their partners. Students could choose to figure out the winner for any number of players from 1 to 25. Then they would add their discoveries to a chart posted on the board (see **table 1**). Their results were to be recorded as follows:

- If the "winner" box was blank, the group would write the winner's position in the circle for that round.
- If a group arrived at an answer that was different from the number already written in the "winner" box, the students also would write this different answer in the "winner" box, signifying disagreement.
- If a group arrived at an answer that was the same as a number in the "winner" box, the students would write this answer in the "confirmation" box, indicating that the group agreed with one of the answers already recorded.

TABLE 1

Students record their results.

Number of Players	Winner	Confirmation
1	1	1 1 1 1
2	1	1 1 1 1 1 1
3	3	3 3 3 3
4	1	1 1 1
5	3, 5	3 3 5 3 3 3
6	5	5 5 5
7	5, 7	7 7
8	1	1 1 1
9	7, 3	3 3 3 3 3
10	5, 1	5 5 5 5
11	7	7 7 7
12	9	9 9
13	11	11 11 11
14	13	13 13 13 13 13 13
15	3 , 15, 11	3 3 15 11 15 15 15
16	1	1 1 1 1 1
17	11, 5, 3, 7, 15	3 3 3 11 3
18	5, 7	5 5 5
19	7	7 7 7
20	9	9 9
21	11	11 11
22	17, 13, 5	13 13
23	21, 15	15 15
24	21, 17, 9, 1	17 9 1 9 9 9 17 17 17 1 17 17
25	7, 19, 15	19 19 19 19 19

Some groups checked the chart to see which numbers of players needed answers and solved for those, while other groups simply chose player numbers without checking the results on the chart. Some partners worked together to solve problems; others worked separately and then compared their answers. Students also were able to gauge their accuracy based on the results that other groups were posting on the class chart. As the students filled in the chart, I did not comment on their accuracy or ask them to recheck their answers if I noticed they had made a mistake. I allowed them to work independently.

With rare exceptions, groups used the pencil-and-paper method in a similar fashion, writing out lists of numbers and crossing them out. One group took a different approach. “Every time you go around the circle,” one student explained to me, “half the players get out. That’s like dividing by 2. So we are using a calculator and dividing the starting number by 2 over and over to see if we can figure out the winner.” His partner was checking the calculator’s answers with pencil and paper. When I asked if this process seemed to reveal the winner, the students acknowledged that it did not, but they were not yet done experimenting. “What does your method tell you so far?” I asked. After some thought, one of the students said, “We can tell how many times around the circle you have to go to get a winner.” Eventually, after more unsuccessful attempts to reveal a winner, the two students abandoned their method. Clearly, though, they had considered the problem from a unique angle, which illuminated their thought processes regarding the problem. The two students later shared their method with the rest of the class during a discussion.

Periodically, I stopped the class and gave an update of our progress. I asked the class what we should do about the disagreements with some of the winning numbers. After some debate and a class vote, the students decided that at least three groups had to arrive at the same answer in order for the class to feel confident that the winner for a given number of players had been determined. With this decision guiding us, I pointed out rounds that had yet to be addressed, did not have the required number of confirmations, or were closely divided among different answers and needed additional checking. I listed these numbers on the board and asked groups to concentrate on the numbers in order to complete our chart.

Once disagreements had been resolved, we crossed out the incorrect answers. During a discussion of our efforts to fill in the chart, students noted that it was “easier to lose your place” when crossing out numbers in the larger circles, which caused an increased amount of errors. The class also was struck by the fact that different groups arrived at the same wrong answers for the same problems. By the end of the second day, we had finished our chart.

Looking for Patterns

The next day, I rewrote a “clean” copy of our chart for 1 to 60 players, filling in the winners we had discovered for 1 to 25 players (see **table 2**). During a group discussion, students looked for patterns on the chart and explained their observations to the class. Students found and shared the following patterns, which I recorded on the board for later reference:

- "Winning numbers go in order, getting bigger, skipping by two. They are all odd numbers."
- "No even numbers ever win. They always get knocked out the first time around the circle."
- "Player 1 wins when this many players are in the circle: 1, 2, 4, 8, 16. Starting with 1, you double the number [or multiply it by 2] to get the next time Player 1 wins."
- "Player 3 wins when this many players are in the circle: 3, 5, 9, 17. You start with 3 and then double the number and subtract 1." [3 doubled = 6, 6 – 1 = 5; 5 doubled = 10, 10 – 1 = 9.]
- "The last person in the circle is the winner when this many players are in the circle: 1, 3, 7, 15. You start with 1 and then double the number and add 1." [1 doubled = 2, 2 + 1 = 3; 3 doubled = 6, 6 + 1 = 7.]
- "Another way to figure out when the last player wins is that if you start with 1, you add 1/2 and then multiply by 2." [1 + 1/2 = 1 1/2; 1 1/2 × 2 = 3; 3 + 1/2 = 3 1/2; 3 1/2 × 2 = 7.]
- "Another way to figure out when the last player wins is that if you start with 1, you add 2 to get the next number, then add 4 to get the next number, then add 8 to get the next number. You keep doubling the number that you add to the answer before it." [1 + 2 = 3; 3 + 4 = 7; 7 + 8 = 15.]

Using the patterns that students had discovered, the class made predictions about the winners of higher rounds. Based on the "Player 1" pattern, the class predicted that the next two times Player 1 would win were in rounds 32 and 64. We then predicted that the next two times Player 3 would win were in rounds 33 and 65. Finally, we decided that all three of the "last player" patterns indicated that the next time the last player in the circle would win was in round 31. We added all these numbers to our chart, putting a question mark by them so we knew that they were predictions that needed to be confirmed. I pointed out to the class that, through the application of different problem-solving strategies, we had generated our predicted answers using significantly less time and effort than we had used when creating our original list of answers. This fact demonstrated the power of effective problem solving.

Students also noticed that there were "lists" of odd numbers on the chart, each of which started with 1 and increased until, at different points, the list "stopped" and "reset" to 1, at which point another ascending, odd-numbered list began. I asked the class to figure out when a given list would stop and reset to 1. The class made a number of predictions, using our posted chart of winners, the written list of patterns, and pencil and paper.

Many students noted that the pattern reset to 1 only on even-numbered rounds. Others realized that the pattern reset to 1 on the round after the last player in the circle had won (see **table 3**).

Another student counted how many numbers, or rounds, were in each list before it reset to 1. Others in the class excitedly picked up on the pattern: One number is on the first list (1), and a new list starts over at 1. Two numbers are on the next list (1, 3), and a new list starts again, this time with four numbers on it (1, 3, 5, 7).

Using this pattern, we extended our chart beyond 25 players and predicted that Player 31 would win with 31 players in the circle. Then the winner would reset to Player 1 with 32 players in the circle (see **table 4**).

We confirmed this prediction by applying three of the other patterns or rules that we had discovered:

1. When Player 1 will win
2. When the last person in the circle will win
3. The relationship between the last player winning

TABLE 2

The corrected chart, with room for greater numbers of players

Number of Players	Winner		Number of Players	Winner
1	1		31	
2	1		32	
3	3		33	
4	1		34	
5	3		35	
6	5		36	
7	7		37	
8	1		38	
9	3		39	
10	5		40	
11	7		41	
12	9		42	
13	11		43	
14	13		44	
15	15		45	
16	1		46	
17	3		47	
18	5		48	
19	7		49	
20	9		50	
21	11		51	
22	13		52	
23	15		53	
24	17		54	
25	19		55	
26			56	
27			57	
28			58	
29			59	
30			60	

TABLE 3

One prediction of when the winning number will "reset" to 1

Players	Winner	Result
1 2	1 1	Last player wins Player 1 wins
3 4	3 1	Last player wins Player 1 wins
7 8	7 1	Last player wins Player 1 wins
15 16	15 1	Last player wins Player 1 wins
31 32	31? 1?	

? = Prediction

TABLE 4

Predicting winners in future rounds

Number of Players	Winner	Numbers on List
1	1	1
2 3	1 3	 2
4 5 6 7	1 3 5 7	 4
8 9 10 11 12 13 14 15	1 3 5 7 9 11 13 15	 8
16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31	1 3 5 7 9 11 13 15 17 19 21? 23? 25? 27? 29? 31?	 16 ?
32	1?	

? = Prediction

and Player 1 winning

All our ideas, rules, and patterns were beginning to converge. We could predict the winner for a given number of players with one of the patterns we had discovered and confirm our prediction with other patterns. In the above instance, we made and checked a prediction with four different patterns.

Our work so far had taught my class valuable lessons about the nature of problem solving. Taking different approaches to the same problem was not only possible but also enabled new insights and confirmation of ideas and answers to occur. Students looking at the same data analyzed it in myriad ways, as shown by the different descriptions of, and means of calculating, the various number sets. Also, students who had different abilities or sets of mathematical knowledge viewed our pattern-seeking in different ways. Some students used multiplication, while others used repeated addition. Some students added or multiplied fractions, while others used only whole numbers. Some students used mental computation, while others used pencil and paper or calculators.

Using the Patterns

During the last two days of the activity, I asked the class to complete the following tasks:

1. Fill out a chart of winners for 1 to 60 players.
2. Make a list of the first fifteen rounds in which Player 1 wins.
3. Make a list of the first fifteen rounds in which Player 3 wins.
4. Make a list of the first fifteen rounds in which the last player wins.
5. Bonus Challenge: Which player will win when there are 511 players? 2049 players? 4096 players?

I reminded my class that good problem solvers try to be thoughtful, careful, and efficient. I asked the class to give me an example of how *not* to proceed. One student, with most of the other students in agreement, said, "You shouldn't write out all the numbers from 1 to 4096 and then cross them all out. It would take forever and you would probably make a mistake and have to start over, and your hand would hurt." (For a later math challenge, I asked students to figure out how long it actually would take to do this, assuming that writing a number or crossing it out would take one second.) "It would be easier to use the patterns we made up to figure out the winner," the student added.

Another student said, "We could get 511 people to stand in a circle and play the game, but it would take a long time and how could we find that many people?"

While many students thought that this method would be "really awesome," they acknowledged the impracticality of it. I agreed and asked the students to apply patterns to solve the problems, instead of using a lengthy elimination process.

I encouraged the students, working with partners or on their own, to use any means or tools to complete the challenges. Students were permitted to use calculators if they thought they would be helpful. As they worked, I asked the students how they were proceeding, how they had calculated a given answer, and what information on their chart, or what patterns, they had employed to find an answer.

Students approached the tasks in various ways. Some filled out the chart in order. Others filled in all the 1s, then all the last players, then all the 3s. Students selected different patterns to use while making a list of when the last player would win. All the students used patterns to calculate when Player 1 would win, but not everyone used those results to complete the Player 3 list. Realizing that Player 3 always won on the round after Player 1, some students created the Player 3 list simply by increasing by one each of the numbers on the Player 1 list; that is, 2, 4, 8 becomes 3, 5, 9. Others did not discover this method and calculated a Player 3 list from scratch without referring to the completed Player 1 list. When time permitted, students who had finished all the tasks tackled bigger challenges, such as determining the winner for rounds 65,539 or 131,072.

Conclusion

This project started out as a fun game of elimination that I played with my class when we found ourselves with an extra five or ten minutes. Throughout the week that we played the game, however, my class was totally immersed in it. Mathematically, it employed many of the NCTM Content Standards—Number and Operations, Algebra (patterning), and Data Analysis—as well as the Process Standards, most notably Problem Solving but also Reasoning and Proof, Communication, and Connections. The activity required my students to employ a range of problem-solving methods, such as making charts, diagrams, and lists; looking for patterns; doing a simpler problem; working backward; guess and check; and logical reasoning. It also allowed my students to make discoveries independently, in small groups, and as a class. The activity challenged all my students, from those who struggled with mathematics to those who were quite advanced.

References

Burns, Marilyn. *Math for Smarty Pants*. New York: Scholastic, 1982.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, Va.: